1. A fair coin is tossed until a head appears. Let $X$ denote the number of tosses required.
   
   1. Find the probability distribution of $X$.
   2. Find the mean and variance of $X$.
   3. Write a GAUSS program to find (approximately) the probability that $X = 1, 2, ..., 5$.

2. An urn contains balls numbered 1, 2, and 3. First a ball is drawn from the urn and then a fair coin is tossed the number of times given by the number written on the ball that is drawn. Define the random variable $X$ as the total number of Heads obtained.

   1. Find the expected value of $X$.
   2. Write a GAUSS program to “answer” the same question. [Of course, in any given run of $N$ trials the analytic and numerical answers will differ, though they should be close “on average.”]

3. (MGB VI.3)

   1. What is the probability that the two observations from a random sample of 2 from a Uniform distribution on $[0,1]$ (i.e., $U[0,1]$), will not differ by more than .5?
   2. What is the probability that the mean of a sample of two observations from $U[0,1]$ will be between .25 and .75?

4. (MGB VI.30) For a random sample of size 2 from $N(0,1)$, find the distribution of the range.
5. Let $X$ be distributed as a uniform random variable on the interval $[0, a]$, where $a$ is an unknown parameter. You have access to a random sample of size $n$.

1. Let $Y_1, ..., Y_n$ denote the order statistics associated with the sample. Show that the maximum likelihood estimator of $a$ ($\hat{a}_1$) is equal to $Y_n$. Derive the distribution of the estimator. Is the estimator unbiased? Is it consistent?

2. Define the estimator $\hat{a}_2 = 2\overline{X}_n (= 2\overline{Y}$ as well) Derive the distribution of this estimator. Is it unbiased? Is it consistent?

3. Define at least one other estimator of $a$ that is unbiased. Discuss the distribution of the estimator you have proposed.

4. If you had to choose one estimator to use in a given application, which would it be? Would your choice of estimator depend on $n$? Defend your choice.

6. (Based on MGB VII.6) Observations $X_1, X_2, ..., X_n$ are drawn (independently) from Normal distributions with the same mean $\mu$ but different variances $\sigma^2_1, \sigma^2_2, ..., \sigma^2_n$.

1. Is it possible to estimate all of the parameters based on one sample realization (that contains $n$ draws)?

2. Continue to assume that the $\sigma^2_i, i = 1, ..., n$, are unknown. Is the sample mean $\overline{X}$ an unbiased estimator of $\mu$? Is it consistent?

3. Say that you know the values of the variances. Derive the maximum likelihood estimator of $\mu$. 

2