Econometrics I

Fall 2000

Assignment 5

Today's Date: October 11
Due Date: October 17

Please show all of your work and clearly indicate your final response to each question.

1. Let a random variable Y be normally distributed in the population with mean μ (the value of which is unknown to you) and variance 1. You are to draw a sample of size 3 from the population and, on the basis of the sample observations, you are to estimate μ. Consider the following three possible estimators:

   - $\hat{\mu}_1 = (Y_1 + Y_2 + Y_3)/3$
   - $\hat{\mu}_2 = Y_{(2)}$, where $Y_{(2)}$ denotes the second largest value observed in the sample.
   - $\hat{\mu}_3 = (Y_{(1)} + Y_{(3)})/2$, which is the average of the smallest and largest values in the sample.

   (a) Determine whether each of the estimators is unbiased.

   (b) For all of the unbiased estimators, determine the variances of the estimators. On the basis of your results, which estimator do you prefer?

   (c) Are each of the three estimators normally distributed? If not, what accounts for the normality of some and the nonnormality of others?

2. Let X be a single draw from a Normal distribution with mean 0 and variance $\theta (= \sigma^2)$.

   (a) Is X a sufficient statistic?

   (b) Is $|X|$ a sufficient statistic?

   (c) Is X^2 an unbiased estimator of θ?

3. Let $X_1, ...X_N$ be a random sample from the density $f_X(x;\theta) = \theta x^{-2}\chi[x \geq \theta]$, with the unknown $\theta > 0$. Is $X_{(1)} = \min\{X_1, ..., X_N\}$ a sufficient statistic?

4. Let $X_1, ..., X_N$ be a random sample from the population distribution $N(\theta, \theta^2)$, $\theta \in R^1$.

   (a) Does there exist a scalar sufficient statistic for θ?

   (b) Find a two-dimensional sufficient statistic.

   (c) Is the sample mean $\bar{X}$ a Minimum Variance Unbiased Estimator of θ? [Hint: Find an unbiased estimator of θ based on the sample variance - call it T. Then find a constant a to minimize $VAR\{a\bar{X} + (1-a)T\}$.].

5. Determine the condition (on the a_i) under which $\sum_{i=1}^{N} a_i X_i$, where $(X_1, ..., X_N)$ is a random sample, is an unbiased estimator of $E(X)$.

6. Let X follow a uniform distribution on the interval $[0, \theta]$, $\theta > 0$. You have access to a random sample $[X_1, ..., X_N]$..

   (a) Show that $X_{(N)}$ is a sufficient statistic.

   (b) Show that $2\overline{X}_N$ is an unbiased estimator of θ, and find its variance.

   (c) Let $N = 2$. Compute the expected value of the estimator $2\overline{X}_N$ conditional on $X_{(2)}$. Is this estimator unbiased? Is its variance smaller than $2\overline{X}_N$ (refer to the Rao-Blackwell theorem).

   (d) For general N, show that the MVUE estimator of θ is $\frac{N+1}{N} X_{(N)}$.