Econometrics I

Final Examination

Fall 1999

Please answer all the questions and show all of your work. If you think a question is ambiguous, clearly state how you interpret it before providing an answer. Good luck!

1. Consider the estimation of the following simple regression model,

$$y_i = \beta x_i + \varepsilon_i,$$

where $x_i = X_i - N^{-1}\sum_{i=1}^{N} X_i$, and the sequence $X_1, ..., X_N$ is assumed to be fixed. The disturbance term ε_i has independently distributed with mean 0 and variance $\sigma_i^2,\ i = 1, ..., N$..We assume that $\lim_{N\to\infty} N^{-1}\sum_{i=1}^{N} x_i^2 = Q$, $0 < Q < \infty$.

   1. Find the OLS estimator for β, .denoted $\hat{\beta}$. Prove whether or not $\hat{\beta}$ is consistent. Find the standard error of $\hat{\beta}$.
   2. Assuming that the $\sigma_1^2, ...\sigma_N^2$ are known, write down the Generalized Least Squares estimator of β, denoted by $\tilde{\beta}$. Show that $\tilde{\beta}$ is a consistent estimator of β. Find the standard error of $\tilde{\beta}$.
   3. If both $\hat{\beta}$ and $\tilde{\beta}$ are consistent estimators of β, determine which one is more efficient if it is possible to do so. Are both estimators linear functions of the data? Is one of them the best linear unbiased (not only consistent) linear estimator of β? If so, which one?
   4. Let ε be normally distributed. Write the log likelihood function for the sample. If they exist, determine the sufficient statistic(s) for the maximum likelihood estimator of β. Based on the sufficient statistics, what can you conclude about the efficiency properties of $\tilde{\beta}$?
   5. Now assume that the σ_i^2 are unknown. Find an unbiased estimator of $\sigma_i^2,\ i = 1, ..., N$, if one exists. If an unbiased estimator exists, is it consistent? Give some intuition as well as a rigorous argument.
   6. Assume that the conditional variance of each observation is given by

$$\sigma_i^2 = \delta z_i,$$

   where z_i is an observed individual characteristic and $z_i > 0$ for all i and $\delta > 0$ is a known parameter. Can you find an unbiased estimator of $\sigma_i^2,\ i = 1, ..., N$? If so, what is it? If not, why not?Determine whether or not the random variable $\sqrt{n}(Y_n - \theta)$ has a degenerate asymptotic distribution [recall that a distribution is degenerate if it has zero variance].

7. Let $\overline{X}_n \equiv n^{-1} \sum_{i=1}^{n} X_i$. Determine the mean and variance of $\overline{X}_n$.
8. Find the asymptotic distribution of $2\overline{X}_n$.
9. Compare the asymptotic variance of Y_n and $2\overline{X}_n$ [that is, the variances of the asymptotic distributions]. If it is possible to rank them, which is smallest?

2. (18 points) Let X and Y be two independent random variables. The random variable X is exponentially distributed with density $f_X(x) = \alpha \exp(-\alpha x)$ for $x > 0$ and $\alpha > 0$. The random variable Y is distributed as a chi-square with k degrees of freedom. The moment generating function of Y is given by $m_Y(t) = (1 - 2t)^{-k/2}$ for $t < .5$.

   1. Find the m.g.f. of X, $m_X(t)$.
   2. Form the random variable $Z = X + Y$. Using $m_X(t)$ and $m_Y(t)$, find the mean and variance of Z.

3. (28 points) In each of two subpopulations the random variable X is distributed according to a power distribution, that is

$$F_{X|\alpha}(x|\alpha) = \begin{cases} 0 & x < 0 \\ x^{\alpha} & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}, \ \alpha > 0.$$

   The parameter α takes the value 1 in subpopulation 1 and takes the value 2 in subpopulation 2. The population is evenly divided between the two subpopulations.

   1. Find the density of X in the population.
   2. A population member can be characterized in terms of their value of X and the subpopulation to which they belong, d [which equals 0 if the individual belongs to subpopulation 1 and equals 1 if the individual belongs to subpopulation 2]. Given knowledge of d, find the function $h^*(d)$ that solves:

$$h^*(d) = \arg\min_{h(d)} E_{X|d}(X - h(d))^2,$$

      where $E_{X|d}(X - h(d))^2$ denotes the expectation of the function $(X - h(d))^2$ taken with respect to the conditional distribution of X given d.
   3. Reconsider the prediction problem above under the restriction that $h(d)$ be a linear function of d. Is the solution the same as above? Why or why not?
   4. How much of the total variation in X is "attributable" to variability in the conditional expectations of $X|d$? [Use the Analysis of Variance decomposition.]

4. (18 points) Let $\overline{X}_n$ denote the sample mean of a random sample consisting of n draws from an exponential distribution with parameter δ, i.e., $f_X(x) = \delta \exp(-\delta x)$ for $x > 0$ and $\delta > 0$.

   1. Find the asymptotic distribution of $\overline{X}_n$.
   2. The maximum likelihood estimate of δ is given by $\hat{\delta} = \overline{X}_n^{-1}$. Find the asymptotic distribution of $\hat{\delta}$.