

Game-Theoretic Perspectives on Coalition Formation

Debraj Ray
New York University

Kyle Hyndman, Hideo Konishi, Rajiv Vohra . . .

Francis Bloch, Armando Gomes/Philippe Jehiel, Eric Maskin . . .

Examples

customs unions

oligopoly cartels

cross-national environmental agreements

credit and insurance cooperatives

R&D consortia

political party formation

legislative bargaining

group or coalition: a collective of individuals who agree to write a binding agreements with one another; cannot do so with nongroup members.

fundamental elements: cooperation “within”, noncooperation “across”

Predicting coalition structure

Studying efficiency (or lack thereof)

A Brief History of Cooperative Game Theory

(the von Neumann-Morgenstern legacy)

Start with a game: action sets, payoff functions etc.

Construct the *characteristic function*.

An Example: Cournot Oligopoly

Unit cost c .

An Example: Cournot Oligopoly

Unit cost c . $P = A - bx$.

Cournot payoffs with set N of n firms:

$$\frac{1}{b} \left(\frac{A - c}{n + 1} \right)^2$$

An Example: Cournot Oligopoly

Unit cost c . $P = A - bx$.

Cournot payoffs with set N of n firms:

$$\frac{1}{b} \left(\frac{A - c}{n + 1} \right)^2 = \left(\frac{1}{n + 1} \right)^2$$

(normalizing $\left(\frac{A - c}{n + 1} \right)^2$ to 1).

Characteristic function: $v(N) = 1/4$, $v(S) = 0$ all other S .

Messrs. von Neumann and Morgenstern are *not* comfortable with this!

“The desire of the coalition $-S$ to harm its opponent, the coalition S , is by no means obvious. Indeed, the natural wish of the coalition $-S$ should not be so much to decrease the expectation value ... of the coalition S as to increase its own expectation value. These two principles would be identical ... when Γ is a zero-sum game, but it need not be at all so for a general game ...” (p. 540)

but go ahead anyway:

“Inflicting losses on the adversary may not be directly profitable in a general game, but it is the way to put pressure on him. He may be induced by such threats to pay a compensation, to adjust his strategy in a desired way, etc. ... It must be admitted, however, that this is not a justification of our procedure — it merely prepares the ground for the real justification which consists of success in examples.” (p. 541)

Discuss Cournot example for 2 and 3 players.

Recall Cournot payoff with n firms:

$$\left(\frac{1}{n+1} \right)^2$$

Monopoly profits $1/4$. Someone getting no more than $1/12$.

One firm breaks off, induces the duopoly, gets $1/9$.

But it doesn't stop there.

Lessons from the Cournot example

1. Intercoalitional interaction

Good shorthand is the *partition function*, built from intercoalitional Nash equilibria:

$V(S, \pi)$, where $S \in \pi$.

2. Prediction

(need to forecast π)

3. Farsightedness

(need to look ahead to the “ultimate” outcome)

Literature splits on two sub-approaches:

Coalitions as fundamental units:

Literature splits on two sub-approaches:

Coalitions as fundamental units: based on blocking, dominance, etc.

Chwe (1994), Haeringer (2004), Huang-Sjöström (2003), Konishi-Ray (2003), Maskin (2003), Ray-Vohra (1997), Ray (1989), Rosenthal (1972), Thrall and Lucas (1963), . . .

Individuals as fundamental units:

Literature splits on two sub-approaches:

Coalitions as fundamental units: based on blocking, dominance, etc.

Chwe (1994), Haeringer (2004), Huang-Sjöström (2003), Konishi-Ray (2003), Maskin (2003), Ray-Vohra (1997), Ray (1989), Rosenthal (1972), Thrall and Lucas (1963), . . .

Individuals as fundamental units: based on a given extensive form, typically with bargaining

Bloch (1996, 1997), Chatterjee-Dutta-Ray-Sengupta (1993), Gomes-Jehiel (2003), Hyndman-Ray (2004), Okada (2000), Rubinstein (1982), Ray and Vohra (1999, 2001), Seidmann and Winter (1998)
. . .

Game of Coalition Formation

1. A set N of players.

Coalition: any nonempty subset of N .

Coalition structure: partition π of N into coalitions.

2. A set X of states.

At minimum x is (π, \mathbf{u}) , where π is a partition and \mathbf{u} is a payoff vector.

$\mathbf{u}_S \in V(\pi, S)$, the *partition function*.

3. *An approval correspondence* $F_S(x)$.

Given x , tells you all states coalition S can unilaterally “implement” or “approve” .

Can use this to think about different kinds of agreements.

4. *A protocol for proposals and responses.*

This is where we depart from “coalitions-as-units” approach.

Two simple protocols:

first rejector

4. *A protocol for proposals and responses.*

This is where we depart from “coalitions-as-units” approach.

Two simple protocols:

first rejector *random*

At x , a proposer proposes x' to S

(S must be able to approve x' : $x' \in F_S(x)$).

S must accept the proposal, otherwise state unchanged to next period.

5. *A set of payoff functions.*

$$\sum_{t=0}^{\infty} \delta_i^t u_i(x(t))$$

[Take expectations if $\{x(t)\}$ not deterministic.]

“Solving” These Games

Strategy:

“Solving” These Games

Strategy: tells you what to do each time you’re proposer or responder.

Payoff-relevant material:

The going state, if you’re the proposer

The going state *and* the proposal, if you’re the responder

Markov strategies and Markov equilibrium

The Case of Irreversible Agreements

Coalition forms; exits

Stationary payoff flow once all coalitions form

Two Assumptions

Anonymous Partition Functions

Write as $v(s, \mathbf{n}) > 0$ for coalition of size s and numerical structure \mathbf{n} .

E.g., in Cournot example $v(s, \mathbf{n}) = 1/(|\mathbf{n}| + 1)^2$.

“Rejector-Proposes” Protocol

First person to reject previous offer proposes; otherwise select one at random.

An Algorithm

Substructure π : collection $\mathbf{n} = (n_i)$ such that

$$K(\mathbf{n}) \equiv \sum n_i < n.$$

Empty substructure also included by convention; denoted \emptyset .

Want to assign a number $t(\mathbf{n})$ to every substructure.

Then generate a (numerical) coalition structure by applying $t(\cdot)$ recursively, starting from \emptyset .

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion.

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion. Suppose $t(\mathbf{n})$ already defined whenever $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$.

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion. Suppose $t(\mathbf{n})$ already defined whenever $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$. For any such \mathbf{n} , let

$$c(\mathbf{n}) \equiv (\mathbf{n}$$

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion. Suppose $t(\mathbf{n})$ already defined whenever $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$. For any such \mathbf{n} , let

$$c(\mathbf{n}) \equiv (\mathbf{n}.t(\mathbf{n}))$$

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion. Suppose $t(\mathbf{n})$ already defined whenever $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$. For any such \mathbf{n} , let

$$c(\mathbf{n}) \equiv (\mathbf{n}.t(\mathbf{n}).t(\mathbf{n}.t(\mathbf{n})))$$

Step 1. If $K(\mathbf{n}) = n - 1$, set $t(\mathbf{n}) \equiv 1$.

Step 2. Recursion. Suppose $t(\mathbf{n})$ already defined whenever $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$. For any such \mathbf{n} , let

$$c(\mathbf{n}) \equiv (\mathbf{n}.t(\mathbf{n}).t(\mathbf{n}.t(\mathbf{n})).\dots).$$

Step 3. For $K(\mathbf{n}) = m$, let $t(\mathbf{n})$ be the *largest* integer in $\{1, \dots, n - m\}$ that maxes

$$\frac{v(t, c(\mathbf{n}.t))}{t}.$$

Step 4. When finished, t defined for every substructure (including \emptyset).

Let $\mathbf{n}^* \equiv c(\emptyset)$.

Equilibrium is *standard* if at every stage the proposer makes — with positive probability — an acceptable proposal that includes herself in coalition.

Proposition. *For discount factors close enough to 1, any standard equilibrium must generate the numerical coalition structure \mathbf{n}^* .*

1. When does a standard equilibrium exist (so that \mathbf{n}^* is always a prediction)?
2. When is \mathbf{n}^* the *only* predicted outcome?

Question [1], existence.

Define

$$a(\mathbf{n}) \equiv \frac{v(t(\mathbf{n}), c(\mathbf{n}.t(\mathbf{n})))}{t(\mathbf{n})}.$$

“Algorithmic average worth” at \mathbf{n} .

Algorithmic average worth is *weakly nonincreasing* if

$$a(\mathbf{n}) \geq a(\mathbf{n}.t(\mathbf{n}))$$

for every substructure \mathbf{n} such that $\mathbf{n}.t(\mathbf{n})$ is also a substructure.

Proposition. *A standard equilibrium exists for discount factors close to one if and only if algorithmic average worth is weakly nonincreasing.*

Question 2, uniqueness.

At \mathbf{n} , say t is a *restricted maximizer* if it maxes

$$\frac{v(t, c(\mathbf{n}.t))}{t}$$

subject to some upper bound on t .

Algorithmic average worth is *strongly nonincreasing* if

$$a(\mathbf{n}) \geq a(\mathbf{n}.t)$$

for every substructure \mathbf{n} and restricted maximizer t such that $\mathbf{n}.t$ is also a substructure.

Proposition. *If algorithmic average worth is strongly nonincreasing, then for discount factors close to one every equilibrium is standard, and \mathbf{n}^* is the unique numerical coalition structure .*

Implications

1. *Does the average worth condition “usually” hold?*

Holds in every application we’ve considered.

Of course, can cook up partition functions where it fails.

$$\mathbf{v}(4, 1) = (6, 1), \quad \mathbf{v}(3, 2) = (3, 8), \quad \mathbf{v}(2, 1, 1, 1) = (0.1, 3, 3, 3)$$

$$\mathbf{v}(3, 1, 1) = (10, 0, 0), \quad \mathbf{v}(\pi) = 0 \text{ for all other } \pi$$

Algorithmic average worth assumption satisfied in weak form but not strong.

Has a standard equilibrium with structure $\mathbf{n}^* = (4, 1)$.

Also has a “non-standard equilibrium” with structure $(3, 2)$.

In Cournot example condition satisfied. Generates grand coalition for $n \geq 4$

standalone firms plus a single cartel for $n \geq 5$.

2. *Can the outcome exhibit “full noncooperation” (a singleton coalition structure)?*

Never, unless Nash equilibria are themselves efficient.

Indeed, in symmetric games, inefficiency closely linked with asymmetric coalition structures.

3. *Can one compute, or bound, the “degree” of inefficiency?*

Sometimes.

Number of coalitions relative to number of players

Direct estimate of lost surplus

4. *Can the algorithm be extended to general games (no symmetry)?*

Yes.

Lose the average worth property.

Auxiliary conditions of algorithm delicate, but can be dropped for “generic games” .

5. *Do we take the exact prediction of coalition structure seriously?*

Yes and no.

No: hard to imagine that every coalition thinks through the entire string of implications.

[*Yet:* is the shortsighted prediction necessarily a better one?]

Yes: model reveals how even very simple models can yield very complex predictions.

Public Goods Provision

Coalition formation problem very different from the mechanism design problem.

Complete information — mechanism design trivial

“The simple adaptation of the definition of the core which has proven appropriate for private goods economies may not be suitable with public goods economies . . . However, the task of developing an alternative core definition (or some other formalization of the intuitive notion of social stability) which better recognizes the structure of the public goods problem is a very delicate one . . . [and] proves to be a particularly difficult problem.” (Roberts [1974])

n identical agents (regions, firms, countries ...)

Each region contributes towards (global) pollution control.

z units of control requires cost $c(z)$ (increasing, strictly convex).

Payoff when global control is Z and region contributes z is

$$Z - c(z).$$

Coalition S of regions, size s . Each region i contributes z_i . Then payoff is

$$\left[\sum_{i \in S} z_i + Z_{-i} \right] s - \sum_{i \in S} c(z_i).$$

Max this. Effectively $\max s[sz - c(z)]$.

Let $z(s)$ be solution.

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n})$$

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv$$

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv s_i z(s_i)$$

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv s_i z(s_i) - c(z(s_i))$$

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv s_i z(s_i) - c(z(s_i)) + \sum_{j \neq i} s_j z(s_j)$$

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv \left(s_i z(s_i) - c(z(s_i)) + \sum_{j \neq i} s_j z(s_j) \right) s_i.$$

For instance, quadratic case:

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv \left(s_i z(s_i) - c(z(s_i)) + \sum_{j \neq i} s_j z(s_j) \right) s_i.$$

For instance, quadratic case: $c(z) = \frac{1}{2}z^2$.

Then a coalition of size s maxes $sz - (1/2)z^2$,

Let $\mathbf{n} = \{s_1, s_2, \dots, s_m\}$. Payoff to coalition of size s_i is

$$v(s_i, \mathbf{n}) \equiv \left(s_i z(s_i) - c(z(s_i)) + \sum_{j \neq i} s_j z(s_j) \right) s_i.$$

For instance, quadratic case: $c(z) = \frac{1}{2}z^2$.

Then a coalition of size s maxes $sz - (1/2)z^2$, so $z(s) = s$.

So payoff given by $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Algorithmic worth strongly increasing.

So equilibrium structure uniquely pinned down.

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 2$

	coalition structure	average worth
t=2	{ 2 }	2

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 2$

	coalition structure	average worth
t=2	{ 2 }	2
t=1	{ 1,1 }	1 + 0.5 = 1.5

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 3$

	coalition structure	average worth
t=3	{ 3 }	4.5

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 3$

	coalition structure	average worth
t=3	{ 3 }	4.5
t=1	{ 1, 2 }	0.5 + 4 = 4.5

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 4$

	coalition structure	average worth
t=4	{ 4 }	8

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 4$

	coalition structure	average worth
t=4	{ 4 }	8
t=2	{ 2, 2 }	2 + 4 = 6

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 4$

	coalition structure	average worth
t=4	{ 4 }	8
t=2	{ 2, 2 }	2 + 4 = 6
t=1	{ 1, 3 }	0.5 + 9 = 9.5

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 5$

	coalition structure	average worth
t=5	{ 5 }	12.5

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 5$

	coalition structure	average worth
t=5	{ 5 }	12.5
t=4	{ 4, 1 }	8 + 1 = 9

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 5$

	coalition structure	average worth
t=5	{ 5 }	12.5
t=4	{ 4, 1 }	8 + 1 = 9
t = 2	{ 2, 3 }	2 + 9 = 11

Recall payoff $v(s_i, \mathbf{n}) = \left(\frac{1}{2}s_i^2 + \sum_{j \neq i} s_j^2 \right) s_i$.

Average payoff $\alpha(s_i, \mathbf{n}) = (1/2)s^2 + \sum_{j \neq i} s_j^2$.

coalition size s	2	3	4	5	6	7	8
direct payoff $(1/2)s^2$	2	4.5	8	12.5	18	24.5	32

$n = 5$

	coalition structure	average worth
t=5	{ 5 }	12.5
t=4	{ 4, 1 }	8 + 1 = 9
t = 2	{ 2, 3 }	2 + 9 = 11
t=1	{ 1, 1, 3 }	0.5 + 1 + 9 = 10.5

Grand coalition “returns” again at $n = 8$, then at $n = 13$.

But a direct computation becomes harder.

Step 1. Compute a special sequence T^* of integers.

Of course, depends on model but easy to do given the parameters.

Recursive: start with $1 \in T^*$. Suppose done up to n .

“Decompose” $n + 1$ into (s_1, s_2, \dots, s_m) , using T^* . E.g., if T^* up to $n = 8$ is $\{1, 2, 3, 5\}$, then 9 decomposes into $\{1, 3, 5\}$.

Now compare *grand coalition profits* at $n + 1$ with *subcoalitional profits* of s_1 under $\{s_1, \dots, s_m\}$.

If grand coalition better, include $n + 1$ in T^* and go on to $n + 2$.

If not, exclude $n + 1$ from T^* and go on to $n + 2$.

Of course, this is what you check in the quadratic case:

$$\text{Is } (n + 1)^2 / 2 \geq (1/2)s_1^2 + \sum_{j \neq 1} s_j^2?$$

Include in T^* if “yes”, exclude otherwise.

And that gives you the now-familiar sequence

1 2 3 5 8 13

Of course, this is what you check in the quadratic case:

$$\text{Is } (n + 1)^2 / 2 \geq (1/2)s_1^2 + \sum_{j \neq 1} s_j^2?$$

Include in T^* if “yes”, exclude otherwise.

And that gives you the now-familiar sequence

1 2 3 5 8 13 20 ...

Step 2. “Decompose” any integer n using T^* . Example: if

$$T^* = \{1, 2, 3, 5, 8, 13, 20 \dots\},$$

and $n = 19$,

then $\mathbf{d}(n) = (1, 5, 13)$.

This is the predicted coalition structure when there are n regions!

Some Observations

1. *Complex structure in a simple model.*

Periodic recurrence of grand coalition.

2. *How many coalitions?*

Proposition. *The number of coalitions is bounded above by $\log_2 n + 1$.*

Idea of proof.

If $\mathbf{d}(n) = (n_1, \dots, n_k)$, then $n_i \neq n_j$ for all i, j .

Corollary: In T^* , $s_{i+1} < 2s_i$ as long as $s_i \geq 2$.

3. *How much inefficiency?*

Let $g(m)$ = per-capita surplus generated by closed public goods economy of size m ;

$$g(m) \equiv \max z(m) - c(z(m)).$$

Proposition. *The ratio of equilibrium to total surplus is at least*

$$\frac{4}{3} \frac{g(n/2)}{g(n)}.$$

Apply to quadratic case.

3. *How much inefficiency?*

Let $g(m)$ = per-capita surplus generated by closed public goods economy of size m ;

$$g(m) \equiv \max z(m) - c(z(m)).$$

Proposition. *The ratio of equilibrium to total surplus is at least*

$$\frac{4}{3} \frac{g(n/2)}{g(n)}.$$

Apply to quadratic case. Lower bound on efficiency ratio = $1/3$.

Some More Remarks, Inspired By Example

1. *How important is the particular model of negotiation used?*

sensitivity to average worth the main thing

2. *Example has “additive” externalities. Does that help?*

Yes. “ T^* -algorithm” is an order of magnitude simpler than “general” version.

3. *All binding agreements possible. What happened to the Coase theorem?*

4-region quadratic case.

3. *All binding agreements possible. What happened to the Coase theorem?*

4-region quadratic case. Structure is $\{1, 3\}$.

Average payoffs $\{9.5, 5.5, 5.5, 5.5\}$.

Average worth of grand coalition is 8.

3. *All binding agreements possible. What happened to the Coase theorem?*

4-region quadratic case. Structure is $\{1, 3\}$.

Average payoffs $\{9.5, 5.5, 5.5, 5.5\}$.

Average worth of grand coalition is 8. Room here to make everyone better off.

So full efficiency should “finally” prevail?

3. *All binding agreements possible. What happened to the Coase theorem?*

4-region quadratic case. Structure is $\{1, 3\}$.

Average payoffs $\{9.5, 5.5, 5.5, 5.5\}$.

Average worth of grand coalition is 8. Room here to make everyone better off.

So full efficiency should “finally” prevail? Maybe, but need to pass through intermediate stage!

Transition

Three ways to look at renegotiation

1. *A commitment is a commitment and is very costly to reverse.*

E.g., building environmentally friendly factories from scratch.

[This is the case of irreversible binding agreements.]

Transition

2. *A commitment is possibly costly but reversible.*

E.g., modular pollution-control devices.

Transition

2. *A commitment is possibly costly but reversible.*

E.g., modular pollution-control devices. Efficiency only in an end-state sense.

[This is the case of permanent binding agreements, with renegotiation.]

Transition

3. *A commitment can be reversed, but “with holdups”.*

Especially relevant for situations in which the legal structure is weak (e.g. international customs unions).

[Irreversible agreements may be the best way to model this.]

More tomorrow . . .

Coalition Formation in Real Time

Last lecture ended with possibility of renegotiation

Calls for a model in which coalition formation and payoffs happen side by side, in “real time”.

But there are other considerations that call for a real-time model:

1. Consistency Requirements:

1. Consistency Requirements: Impose the “same restrictions” on blocking coalitions as one does on the original coalition.

2. FarSightedness:

1. Consistency Requirements: Impose the “same restrictions” on blocking coalitions as one does on the original coalition.
2. FarSightedness: Players may care about “ultimate” payoff from a move, not its immediate consequences.
3. Multiple Continuations:

1. Consistency Requirements: Impose the “same restrictions” on blocking coalitions as one does on the original coalition.
2. FarSightedness: Players may care about “ultimate” payoff from a move, not its immediate consequences.
3. Multiple Continuations: e.g., “optimistic” vs. “pessimistic” standards of behavior.
4. Cycles:

1. Consistency Requirements: Impose the “same restrictions” on blocking coalitions as one does on the original coalition.
2. FarSightedness: Players may care about “ultimate” payoff from a move, not its immediate consequences.
3. Multiple Continuations: e.g., “optimistic” vs. “pessimistic” standards of behavior.
4. Cycles: With continued negotiation, same structures may recur again and again.

Coalition formation in real time addresses these in a “natural” way.

E.g.

Coalition formation in real time addresses these in a “natural” way.

E.g. role of the discount factor,

Coalition formation in real time addresses these in a “natural” way.

E.g. role of the discount factor, handling of cycles.

“Blocking” versus “Proposals” approach.

Game of coalition formation:

1. A set N of *players*.
2. A set X of *states*.
3. An *approval correspondence* $F_S(x) \subseteq X$, for $x \in X$.
4. A *proposer protocol* $q_i(x)$.
5. For each i , a one-period payoff function $u_i(x)$ and discount factor $\delta_i \in (0, 1)$.

Permanently Binding Agreements with Renegotiation

Recall notion of a state as (π, u) .

Contemplate move from $x = (\pi, u)$ to $y = (\pi', u')$.

Let S be the set of all players who belong to *different* coalitions under π and π' .

Then, if existing agreements are binding, *all* these players must approve the move ...

...so the approval committee for the move should be at least S .

Might it be bigger than S ?

It might, but there are conceptual difficulties.

Simply assume that the payoff vector of “untouched coalitions” is specified by some exogenous mapping. (Embodied in F_S .)

Two Restrictions

[R.1] (Only for characteristic functions): whenever $y \in F_S(x)$, $u_i(y) = u_i(x)$ for all i with coalitional membership unaffected by the move from x to y .

[R.2] (Unless agreements are irreversible): $F_N(x) = X$ for every x .

(The grand coalition can serve as approval committee for everything.)

The Process

Start from any state x .

Choose proposer i using protocol for that state.

i proposes a (possibly) new state y .

The proposal *must* be made to some S such that $y \in F_S(x)$.

If unanimously approved by S , move to y .

Otherwise stay put at x .

Process continues *ad infinitum*.

The Process

Start from any state x .

Choose proposer i using protocol for that state.

i proposes a (possibly) new state y .

The proposal *must* be made to some S such that $y \in F_S(x)$.

If unanimously approved by S , move to y .

Otherwise stay put at x .

Process continues *ad infinitum*. receive payoffs throughout.

Strategies and Value Functions

Person k as *proposer*: $\mu_k(x, y, S)$.

Person k as *responder*: $\lambda_k(x, y, S)$.

Strategy profile $\sigma =$ full collection $\{\mu_k, \lambda_k\}$.

Each such profile generates *value functions* for every individual:

$$V_i^\sigma(x) =$$

$$V_i^\sigma(x) = u_i(x) +$$

$$V_i^\sigma(x) = u_i(x) + \delta_i$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x)$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S)$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) \prod_{j \in S} \lambda_j(x, y, S)$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) [\Lambda(x, y, S)]$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) [\Lambda(x, y, S) V_i^\sigma(y)]$$

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) [\Lambda(x, y, S) V_i^\sigma(y) + (1 - \Lambda(x, y, S))$$

$$\begin{aligned}
V_i^\sigma(x) &= u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) \left[\Lambda(x, y, S) V_i^\sigma(y) \right. \\
&\quad \left. + (1 - \Lambda(x, y, S)) V_k^\sigma(y) \right]
\end{aligned}$$

σ is an equilibrium if

(a) $\mu_i(x, y, S) > 0$ only if $y \in F_S(x)$ and

$$(y, S) \in \operatorname{argmax}_{(T, z): z \in F_T(x)} [\Lambda(x, z, T) V_i^\sigma(z) + (1 - \Lambda(x, z, T)) V_i^\sigma(x)].$$

(b) $\lambda_i(x, y, S)$ equals 1 if $V_i^\sigma(y) > V_i^\sigma(x)$,

$$\begin{aligned}
V_i^\sigma(x) &= u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) \left[\Lambda(x, y, S) V_i^\sigma(y) \right. \\
&\quad \left. + (1 - \Lambda(x, y, S)) V_k^\sigma(y) \right]
\end{aligned}$$

σ is an equilibrium if

(a) $\mu_i(x, y, S) > 0$ only if $y \in F_S(x)$ and

$$(y, S) \in \operatorname{argmax}_{(T, z): z \in F_T(x)} [\Lambda(x, z, T) V_i^\sigma(z) + (1 - \Lambda(x, z, T)) V_i^\sigma(x)].$$

(b) $\lambda_i(x, y, S)$ equals 1 if $V_i^\sigma(y) > V_i^\sigma(x)$, 0 if opposite inequality holds,

$$V_i^\sigma(x) = u_i(x) + \delta_i \sum_{k \in N} q_k(x) \sum_{y, S} \mu_k(x, y, S) \left[\Lambda(x, y, S) V_i^\sigma(y) + (1 - \Lambda(x, y, S)) V_k^\sigma(y) \right]$$

σ is an equilibrium if

(a) $\mu_i(x, y, S) > 0$ only if $y \in F_S(x)$ and

$$(y, S) \in \operatorname{argmax}_{(T, z): z \in F_T(x)} [\Lambda(x, z, T) V_i^\sigma(z) + (1 - \Lambda(x, z, T)) V_i^\sigma(x)].$$

(b) $\lambda_i(x, y, S)$ equals 1 if $V_i^\sigma(y) > V_i^\sigma(x)$, 0 if opposite inequality holds, lies in $[0, 1]$ if equality holds.

Proposition. Assume X is countable. Then an equilibrium exists.

Mixing may be needed

Open question if X a continuum.

Absorption

Equilibrium induces a Markov process on states.

Partitions X into *recurrence classes* and a set of *transient states*.

Process ultimately enters one of the recurrence classes.

Equilibrium is *absorbing* if each recurrence class has a single payoff vector.

Equilibrium is *globally absorbing* if all recurrence classes have one identical payoff vector.

Notions of Efficiency

Static

x dominates y if $u_i(x) > u_i(y)$ for all i .

x is *static efficient* if there is no y that dominates it.

Dynamic

Equilibrium σ *dynamically efficient* if $\mathbf{v}^\sigma(x) = (v_1^\sigma(x), \dots, v_n^\sigma(x))$ isn't dominated by any sequence of states.

Can apply static efficiency to absorbing states of equilibria to understand “limiting efficiency” .

ok for discount factors close to 1

Characteristic Functions

Proposition. *If the game of coalition formation is derived from a characteristic function, then all equilibria are absorbing.*

Moreover, every recurrence class — which is degenerate — is static efficient.

Related to older papers with myopic adjustment: Jerry Green, Abhijit Sengupta and Kunal Sengupta . . .

[But this model uses farsighted “adjustment” .]

Tighter connection to bargaining models with ongoing renegotiation: Akira Okada, Daniel Seidman and Eyal Winter . . .

[But this model explicitly set in real time, and coalitions can both contract and expand.]

Externalities and Inefficiency

Inefficiency from some state

$$\begin{array}{ll} x_1 = \pi_1 = \{\{1, 2\}, \{3\}\} & u(x_1) = (4, 4, 4) \\ x_2 = \pi_2 = \{\{1\}, \{2\}, \{3\}\} & u(x_2) = (5, 5, 5) \\ x_3 = \pi_3 = \{\{1\}, \{2, 3\}\} & u(x_3) = (0, 10, 10) \end{array}$$

x_1 must be an absorbing state.

Note, however, that x_3 is absorbing, and it is efficient.

Proposition. *In every three-person coalitional game, every equilibrium is absorbing. Moreover, there is always some absorbing payoff vector which is static efficient.*

Inefficiency from every state

$$\begin{array}{ll} x_1 = \pi_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\} & u(x_1) = (4, 4, 4, 4) \\ x_2 = \pi_2 = \{\{1, 2\}, \{3\}, \{4\}\} & u(x_2) = (5, 5, 5, 5) \\ x_3 = \pi_3 = \{\{1, 2\}, \{3, 4\}\} & u(x_3) = (0, 0, 12, 12) \\ x_4 = \pi_4 = \{\{1\}, \{2\}, \{3, 4\}\} & u(x_4) = (1, 1, 0, 0) \end{array}$$

Claim: Every equilibrium of this game exhibits static inefficiency no matter where you start from.

$$\begin{array}{lll} p(x_1, x_1) = 1 & p(x_2, x_1) = \frac{1}{2} & p(x_2, x_3) = \frac{1}{2} \\ p(x_3, x_3) = \frac{1}{2} & p(x_3, x_4) = \frac{1}{2} & p(x_4, x_1) = 1 \end{array}$$

$x_1 = \pi_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$	$u(x_1) = (4, 4, 4, 4)$
$x_2 = \pi_2 = \{\{1, 2\}, \{3\}, \{4\}\}$	$u(x_2) = (5, 5, 5, 5)$
$x_3 = \pi_3 = \{\{1, 2\}, \{3, 4\}\}$	$u(x_3) = (0, 0, 12, 12)$
$x_4 = \pi_4 = \{\{1\}, \{2\}, \{3, 4\}\}$	$u(x_4) = (1, 1, 0, 0)$

Claim: *Every equilibrium of this game exhibits static inefficiency no matter where you start from.*

1. x_3 and x_4 are not absorbing.
2. x_2 absorbing implies x_2 globally absorbing.
3. x_2 cannot be globally absorbing.
4. x_1 absorbing implies x_1 globally absorbing.
5. Every equilibrium is inefficient.

One equilibrium (for δ close to 1):

$$\begin{array}{lll} p(x_1, x_1) = 1 & p(x_2, x_1) = \frac{1}{2} & p(x_2, x_3) = \frac{1}{2} \\ p(x_3, x_3) = \frac{1}{2} & p(x_3, x_4) = \frac{1}{2} & p(x_4, x_1) = 1 \end{array}$$

Another equilibrium: 1 and 2 randomize between x_1 and x_2 , while 3 and 4 randomize between x_2 and x_3 .

Perpetual (but stochastic) cycling. Also inefficient.

Discussion

1. *Thinking about Subadditivity*

the competitive urge

span of control

superadditive covers and the nature of binding agreements

2. Superadditivity

[GCS] For every state $x = (u, \pi)$, there is $x' = (u', \{N\})$ such that $u' \geq u$.

variation on previous example: remove x_1 , add superadditive cover:

$x_2 = \pi_2 = \{\{1, 2\}, \{3\}, \{4\}\}$	$u(x_2) = (5, 5, 5, 5)$
$x_3 = \pi_3 = \{\{1, 2\}, \{3, 4\}\}$	$u(x_3) = (0, 0, 12, 12)$
$x_4 = \pi_4 = \{\{1\}, \{2\}, \{3, 4\}\}$	$u(x_4) = (1, 1, 0, 0)$
$x_5^a = (\{N\}, u^a)$	$u(x_5^a) = (5, 5, 5, 5)$
$x_5^b = (\{N\}, u^b)$	$u(x_5^b) = (0, 0, 12, 12)$
$x_5^c = (\{N\}, u^c)$	$u(x_5^c) = (1, 1, 0, 0)$

For $\delta \simeq 1$, there is still an equilibrium which cycles over $\{x_2, x_3, x_4\}$, causing inefficiency.

However . . .

Proposition. *If GCS holds, every absorbing state must be static efficient.*

Another equilibrium in our example has the property that

$$x_2 \xrightarrow{(1/2)} x_3 \xrightarrow{(1/2)} x_4 \xrightarrow{1} x_5^a$$

(absorption plus efficiency)

Proposition. *If a game exhibits GCS and the payoff frontier for the grand coalition is continuous and concave, every equilibrium must be absorbing — and therefore static efficient.*

3. *Large UpFront Transfers*

[Armando Gomes and Philippe Jehiel]

Coasian versus malevolent role of transfers

$$\begin{array}{ll} x_1 = \pi_1 = \{\{1, 2\}, \{3\}\} & u(x_1) = (4, 4, 4) \\ x_2 = \pi_2 = \{\{1\}, \{2\}, \{3\}\} & u(x_2) = (5, 5, 5) \\ x_3 = \pi_3 = \{\{1\}, \{2, 3\}\} & u(x_3) = (0, 10, 10) \end{array}$$

In this example, transfers restore efficiency.

But in this example, transfers destroy efficiency.

$$\begin{array}{ll} x_1 = \pi_1 = \{\{1, 2\}, \{3\}\} & u(x_1) = (15, 10, 0) \\ x_2 = \pi_2 = \{\{1\}, \{2, 3\}\} & u(x_2) = (0, 15, 10) \\ x_3 = \pi_3 = \{\{1\}, \{2, 3\}\} & u(x_3) = (10, 0, 15) \\ x^* = \{\{1\}, \{2\}, \{3\}\} & u(x^*) = (16, 16, 16) \end{array}$$

Remark on three-player games

insert diagram

Characteristic functions and transfers: no paradoxes here

So intimately connected with externalities

Taking Stock

Coalition formation with externalities takes us into completely new ground.

A. Theoretical Concerns

A.1. Blocking versus bargaining

A.2. Self-generation for coalitions

A.3. A full analysis of partition functions without upfront transfers

A.4 Axiomatics a la Maskin

A.5. History-dependent interaction across given groups

B. Applied Concerns

B.1. Networks with externalities (double-generalization of partition functions)

B.2. [to be completed]

4. Temporarily Binding Agreements

Like transfers, has ambiguous effects on efficiency.

Questions and Applications