1 Asset Pricing in Infinite Horizon models

Assume a representative-agent economy with one good. Let time be indexed by \( t = 0, 1, 2, \ldots \). Suppose that there is no uncertainty at time zero and that \( S \) branches stem out of it. Similarly, from each of these branches, \( S \) more branches stem out, and so on. For a given \( t \), label the (cross-section of) nodes as \( \{1, \ldots, S^t\} \) and note that \( S^t \) increases over time.

Our agent’s utility will now be defined on stochastic processes \( c := \{c_t\}_{t=0}^{\infty} \), where \( c_t : S^t \rightarrow \mathbb{R}^+ \) is a random variable, for each \( t \). Similarly, let \( \omega := \{\omega_t\}_{t=0}^{\infty} \) be a stochastic processes describing the agent’s endowments. Let \( 0 < \beta < 1 \) denote the agent’s discount factor.

1.1 Contingent Markets Economy

Suppose that at time zero, the agent can trade in contingent commodities. Let \( p(s_t) \) denote the price (from time zero) of the contingent commodity in state \( s_t \) (of time \( t \)) and \( p := \{p_t\}_{t=0}^{\infty} \) be the whole stochastic process for prices, where \( p_t : S_t \rightarrow \mathbb{R}^+ \), for each \( t \).

Definition 1 \((p^*, c^*)\) is an Arrow-Debreu Equilibrium if

i. given \( p^* \),

\[
\begin{align*}
    c^* &
        \in \text{arg max} \{ u(c_0) + E_0[\sum_{t=1}^{\infty} \beta^t u(c_t)] \} \\
    \text{s.t.} &
        \sum_{t=0}^{\infty} p^*_t (c_t - \omega_t) = 0
\end{align*}
\]

ii. and \( c^* = \omega^* \).

Note that \( c^* = \omega^* \) means \( c_t(s_t) = \omega_t(s_t) \), at each \( s_t \) and each \( t \).

1.2 Financial Markets Economy

Suppose that throughout the uncertainty tree, there are \( J \) 1-period assets available. Now, at each node agents receive endowments and payoffs from the portfolios they carry from the previous node. At the same time, they choose consumption and rebalance their portfolios.

Let \( \theta := \{\theta_t\}_{t=1}^{\infty} \) denote the sequence of portfolios of the representative agent, where \( \theta_t : S_t \rightarrow \mathbb{R}^J \). In particular, let \( \theta_t(s_t) \) denote the portfolio bought at the predecessor node to \( s_t \) that pays at time \( t \) and \( \theta_{t+1} \) denote the portfolio bought at node \( s_t \) that pays at \( t+1 \).
Definition 2 \{c^*, \theta^*, q^*\} is a FM Equilibrium if

i. given \(q^*\), at each state \(s_t\),

\[
(c^*|s_t, \theta^*|s_t) \in \arg \max \{u(c_t) + E_t[\sum_{j=1}^{\infty} \beta^j u(c_{t+j})]\} \\
\text{s.t.} \\
c_{t+j} + q_{t+j}^* \theta_{t+j+1} = \omega_{t+j} + a_{t+j} \theta_{t+j}, \text{ for } \theta_{t+j} \text{ given} \\
\text{for } j = 0, 1, 2, ...
\]

ii. \(c^* = \omega\) and \(\theta^* = 0\).

Pricing “serious” assets. Even though we just defined equilibrium for 1-period assets, by redefining the returns to an asset we can price a big range of assets, such as stocks and options. From the FOC of the agent’s problem, we obtain

\[
q_t(s_t) = E_t \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} a_{t+1} \right) = E_t (m_{t+1} a_{t+1}) 
\]

(1)

or, by defining

\[
R_{t+1}(s_t) = \frac{a_{t+1}}{q_t(s_t)},
\]

we can rewrite the FOC as

\[
1 = E_t \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right). 
\]

(2)

Consider a stock. Its payoff at any node can be seen as the dividend plus the capital gain, that is,

\[
a_{t+1} = q_{t+1} + d_{t+1},
\]

for some exogenously given dividend stream \(d\). By plugging this payoff into equation (1), we obtain the price of the stock at state \(s_t\).

For a call option on the stock, with strike price \(k\) at some future period \(T\), we can define

\[
a_{t+1} = \max\{q_T - k, 0\}.
\]

Define now

\[
m_{t,T} = \frac{\beta^{T-t} u'(c_T)}{u'(c_t)}
\]

and observe that the price of the option is given by

\[
q_t(s_t) = E_t \left( m_{t,T} \max\{q_T - k, 0\} \right),
\]

for \(T,t\). Note how the price of the option changes with time, as we learn information when approaching the execution period \(T\).

In this setup, the risk-free rate is known at time \(t\) and therefore, equation (2) applied to a 1-period bond yields

\[
\frac{1}{R_{t+1}'} = E_t \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} \right).
\]
For future reference, we shall derive two related pricing formulas. First, note that
\[ q_t = E_t(m_{t+1}a_{t+1}) = Cov_t(m_{t+1}, a_{t+1}) + E_t(m_{t+1})E_t(a_{t+1}) \]
\[ = Cov_t(m_{t+1}, a_{t+1}) + \frac{E_t(a_{t+1})}{R_{t+1}}. \]
To derive a conditional-beta equation, start from
\[ 1 = E_t(m_{t+1}R_{t+1}) = Cov_t(m_{t+1}, R_{t+1}) + E_t(m_{t+1})E_t(R_{t+1}) \] (3)
and rearrange to obtain
\[ E_t(R_{t+1}) - R_{t+1} = -\frac{Cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})} = \]
\[ = \frac{Cov_t(m_{t+1}, R_{t+1})}{Var_t(m_{t+1})} \left( -\frac{Var_t(m_{t+1})}{E_t(m_{t+1})} \right). \] (4)

1.3 Conditional MVF and Hansen-Jagannathan bounds

Observe that at this point we could build the mean-variance frontier as long as we replace all expectations by expectations conditional on the information available to the agents at time \( t \). This is an important observation to keep in mind when doing empirical work on the mean-variance frontier.

Using the definition of correlation coefficient, equation (4) yields
\[ E_t(R_{t+1}) - R_{t+1} = -\frac{\rho_t(m_{t+1}, R_{t+1})\sigma_t(m_{t+1})\sigma_t(R_{t+1})}{E_t(m_{t+1})}. \]

By taking absolute values on both sides and realizing that the correlation coefficient is bounded by -1 and 1, we get to
\[ \frac{|E_t(R_{t+1}) - R_{t+1}|}{\sigma_t(R_{t+1})} \leq \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})}. \] (5)

The left-hand side of this equation is known as the Sharpe ratio of rate of return \( R_{t+1} \). This is a relation between an asset’s Sharpe ratio and the moments of a SDF necessary to price it. A first interpretation of equation (5) is as a restriction on the equilibrium returns for a given SDF. The contribution of Hansen and Jagannathan (1991) consists in reading the equation as a restriction on the set of SDFs that can price a given set of returns. Specifically, the question they analyze is what is the set of (mean,var) of the SDFs that is consistent with a given set of asset prices and payoffs? And, related, what is the mean-variance frontier for SDFs? It turns out that the mean-variance frontier for SDFs has the same geometric properties as the mean-variance frontier for returns and, not surprisingly, a similar orthogonal decomposition can be produced for all the minimum-variance SDFs.

As we shall see, the HJ bounds provide a very convenient framework to compare the relative empirical success of the different asset pricing theories.

\(^1\)The answers can be found in pages 95-100 in Cochrane’s book.
1.4 Stock prices as random walks

Recall the asset-pricing equation derived from the agent’s FOC:

\[ u'(c_t)q_t = E_t(\beta u'(c_{t+1})a_{t+1}) . \]  

(6)

Recall that for stocks, we defined \( a_{t+1} = q_{t+1} + d_{t+1} \). Substituting in,

\[ q_t = E_t \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} (q_{t+1} + d_{t+1}) \right) . \]

When should we expect stock prices to follow a random walk? Assume that no dividends are paid and agents are risk neutral. Then, for values of \( \beta \) close to 1 (realistic for short time periods), we have

\[ q_t = E_t(q_{t+1}) . \]

That is, the stochastic process for stock prices is a martingale. Next, for any \( \{\varepsilon_t\} \) such that \( E_t(\varepsilon_{t+1}) = 0 \) at all \( t \), we can rewrite the previous equation as

\[ q_{t+1} = q_{t+1} + \varepsilon_{t+1} . \]

This process is a random walk if and only if \( Var_t(\varepsilon_{t+1}) = \sigma \) is constant over time.

A more important observation is the fact that marginal utilities times asset prices follow a well-known stochastic process. Again under no dividends,

\[ u'(c_t)q_t = \beta E_t(u'(c_{t+1})(q_{t+1} + d_{t+1})) , \]

which is a supermartingale and approximately a martingale for \( \beta \) close to 1.

1.5 Fundamentals-driven asset prices

For assets whose payoff is made of a dividend and a capital gain, FOC dictate

\[ q_t = E_t(m_{t+1}(q_{t+1} + d_{t+1})) , \]

where

\[ m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} . \]

By iterating forward and making use of the Law of Iterated Expectations,

\[ q_t = \lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t,t+j}d_{t+j} \right) + \lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t,t+j}q_{t+j} \right) , \]

where

\[ m_{t,t+j} = \frac{\beta u'(c_{t+j})}{u'(c_t)} . \]
As we shall see, infinite horizon models (with infinitely lived agents) usually satisfy the no-bubbles condition, or

$$\lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t+j} q_{t+j} \right) = 0.$$  

In that case, we say that asset prices are fully pinned down by fundamentals since

$$q_t = E_t \left( \sum_{j=1}^{\infty} m_{t+j} d_{t+j} \right).$$

2 Empirical work overview

2.1 Unconditional moment restrictions

Recall that our basic pricing equation is a conditional expectation:

$$q_t = E_t(m_{t+1} a_{t+1}),$$

where

$$m_{t+1} = \beta u'(c_{t+1}) u'(c_t).$$

(7)

In empirical work, it is easier to test for unconditional moment restrictions. However, taking unconditional expectations of the previous equation implies a much weaker statement about asset prices than equation (7):

$$E(q_t) = E(m_{t+1} a_{t+1}),$$

(8)

where we have invoked the law of iterated expectations. It should be clear that equation (7) implies but it is not implied by (8).

The theorem in this section will tell us that actually there is a theoretical way to test for our conditional moment condition by making a series of tests of unconditional moment conditions.

Define a stochastic process \( \{z_t\}_{t=0}^{\infty} \) to be conformable if for each \( t \), \( z_t \) belongs to the time-\( t \) information set of the agent. It then follows that for any such process, we can write

$$z_t q_t = E_t(m_{t+1} z_t a_{t+1})$$

and, by taking unconditional expectations,

$$E(z_t q_t) = E(m_{t+1} z_t a_{t+1}).$$

This fact is important because for each conformable process, we obtain an additional testable implication that only involves unconditional moments. Obviously, all these implications are necessary conditions for our basic pricing equation to hold. The following result states that if we could test these unconditional restrictions for all possible conformable processes then it would also be sufficient. We state it without proof.

**Theorem 3** If \( E(x_{t+1} z_t) = 0 \) for all \( z_t \) conformable then \( E_t(x_{t+1}) = 0. \)

By defining \( x_{t+1} = m_{t+1} a_{t+1} - q_t \), the theorem yields the desired result.
2.2 The Equity premium puzzle

Tighter Hansen-Jagannathan Bounds. A different derivation of the HJ bounds than the previously obtained yields tighter bounds by exploiting the information contained in the asset data in a more efficient manner.

Let $m$ denote a SDF and consider projecting it on a given vector of payoffs $x$,

$$m = a + x'b + \varepsilon$$

where

$$b = \text{cov}(x, x)^{-1}\text{Cov}(x, m)$$

$$a = E(m) - E(x'b)$$

$$E(\varepsilon) = 0, E(\varepsilon x) = 0.$$

By using the definition of covariance, we can rewrite $b$ as

$$b = \text{cov}(x, x)^{-1}[q - E(x)E(m)].$$

It follows from this last equation that for the set of SDFs with the same given mean, $E(m)$, we can solve for $b$ just by using payoff data. If $b$ is found in this way, note that

$$\text{Var}(m) = \text{Var}(x'b) + \text{Var}(\varepsilon),$$

where the orthogonality between the error and the regressors has been used. As is clear from this equation, for all these SDFs with the same mean, we obtain a lower bound for their variances and, hence, for their standard deviations. That is,

$$\text{Var}(m) \geq \text{Var}(x'b).$$

The Equity Premium Puzzle. Using the two expressions we have for the HJ bounds, we can derive the feasible regions for mean-standard deviation pairs of SDFs and use them to illustrate the equity premium puzzle (and other anomalies). When using only two assets, the tighter bounds give rise to a cup-shaped region (actually, it is the familiar shape from the MVF). The first version of the HJ bounds is in terms of excess returns so it condenses the information of the two assets into a single excess return. The admissible region derived in this case contains the admissible region obtained using the tighter bounds.

In their version of the Mehra-Prescott puzzle, Hansen and Cochrane (1992) construct the two regions for the U.S. economy by using quarterly data on the real value-weighted NYSE portfolio and the 3-month Treasury-bill returns, from 1947 to 1990. Note that doing so requires assuming that the 2-period economy is replicated in a stationary fashion. They then compute sample means and standard deviations implied by representative agent models with power utility functions. They observe that only hopelessly high degrees of risk aversion can

\footnote{This subsection and the next follow Hansen and Cochrane (1992), Asset Pricing Explorations for Macroeconomics, in NBER macro annual. For similar material, see chapter 21 in Cochrane’s book.}
make the SDF constructed with MRS to fall into the admissible region. And even in that case, it only falls into the looser bounds but never into the tighter ones. In the paper they proceed to illustrate the empirical attempts to solve the puzzle. Just a couple follow. Each of the two attempts has achieved only partial success.

**Habit persistence.** Models with utility functions that differ from the power utility of the benchmark model in incorporating some form of habit persistence have a partial success in producing SDFs that fall into the admissible region with more realistically lower risk aversion parameters.

**Borrowing constraints.** The fact that at equilibrium some agents choices are characterized by inequalities enlarges the admissible region and provides another way to alleviate the puzzle. However, the constraints required are implausibly high.

### 3 Bubbles: Santos and Woodford, Ecta 1997

Let $N$ be the set of nodes. Let $s^0$ denote the root of the tree and $s^t$ be any node at $t$. Denote by $s^t - 1$ the single (immediate) predecessor node to $s^t$. Let $s^T | s^t$ denote that $s^T$ is some successor of $s^t$, for $T > t$.

At each node, there are $k(s^t)$ securities traded. Let $H(s^t)$ be the set of agents which are active at node $s^t$. Let $N^h$ be the subset of nodes of the tree at which agent $h$ is allowed to trade. Also, denote by $\overline{N}^h$ the terminal nodes for agent $h$.

**Assn 1.** If an agent $h$ is alive at some non-terminal node $s^t$, she is also alive at all the immediate successor nodes. That is,

$$s^t \epsilon N^h \setminus \overline{N}^h \implies \{ s^{t+1} \epsilon N : s^{t+1} | s^t \} \subset N^h.$$

**Assn 2.** The economy is connected across time and states. This is achieved when at any state there is some agent alive and non-terminal. Formally,

$$\forall s^t, \exists h : s^t \epsilon N^h \setminus \overline{N}^h.$$

Let $q : N \rightarrow R^{k(s^t)}$ be the mapping defining the vector of security prices at each node $s^t$. Similarly, let $d : N \rightarrow R^{k(s^t - 1)}$ denote the vector-valued mapping that defines the dividends (in units of numeraire) that are paid by the assets that pay at node $s^t$. We shall assume that a security can pay in dividends (units of consumption) and in units of assets. Mapping $b : N \rightarrow R^{k(s^t - 1)k(s^t)}$ defines at each node $s^t$ a $k(s^t) \times k(s^t - 1)$ matrix, whose i-th column denotes the vector of assets that are paid at node $s^t$ by asset $i$.

**Assn 3.** Assume $d() \geq 0$ and $b() \geq 0$.

Each of the households alive at $s^0$ enters the markets with an initial endowment of securities $\tilde{z}^h(s^0)$. Therefore, the initial net supply of assets is given by

$$z(s^0) = \sum_{h \in H(s^0)} \tilde{z}^h(s^0).$$
Define the net supply of securities at any node \( s^t \), \( z(s^t) \), recursively by

\[
z(s^t) = b(s^t)z(s^{t-1}).
\]

We shall assume that \( z(s^0) \geq 0 \). This ensures that \( z(s^t) \geq 0 \), at all nodes.

Observe that a portfolio held at the end of \( s^t \), say \( z(s^t) \), will generally pay dividends (and assets) at several future states \( s^{t+1}, s^{t+2}, \) etc. We shall now construct a mapping that will assign to each node the dividends paid by portfolio \( z(s^t) \).

As an intermediate step, let us construct the dividends mapping for a portfolio made of one unit of each asset that can be traded at \( s^t \). To determine the stream of dividends generated by this 'canonical' portfolio at each successor node, define for all \( s^r | s^t \) with \( r \geq t \), the \( k(s^t) \times k(s^t) \) matrix \( e(s^r | s^t) \) by

\[
e(s^r | s^t) = I_k(s^t)e(s^r | s^t) = b(s^r)e(s^{r-1} | s^t),
\]

for all \( s^r | s^t \) and \( r > t \).

Matrix \( e(s^r | s^t) \) is just a counter of the asset payoffs at \( s^r \) of the canonical portfolio. Now, for all \( s^r | s^t \) with \( r > t \), define

\[
x(s^r | s^t) = d(s^r)e(s^{r-1} | s^t).
\]

This is a \( k(s^t) \) vector \( x(s^r | s^t) \) and its i-th component is the dividends that the i-th asset of the canonical portfolio produces at node \( s^r \), for \( i = 1, \ldots, k(s^t) \). Finally, the dividends mapping for the vector \( z(s^t) \) is simply \( x(s^r | s^t)z(s^t) \), for each node \( s^r | s^t \) with \( r > t \).

Define an asset \( j \) to have finite maturity if there exists a \( T \) such that \( e(s^r | s^t) = 0 \), for all \( s^r | s^t \) and \( r \geq T > t \).

**Endowments.** At each node \( s^t \), each households in \( H(s^t) \) has an endowment of numeraire good of \( w(s^t) \geq 0 \). We shall assume that the economy has a well-defined aggregate endowment

\[
w(s^t) = \sum_{h \in H(s^t)} w^h(s^t) \geq 0
\]

at each node \( s^t \). Taking into account the dividends paid by securities in units of good, the aggregate good supply in the economy is given by

\[
\bar{w}(s^t) = w(s^t) + d(s^t)z(s^{t-1}) \geq 0.
\]

**Utility function.**

\[
U(c) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_{st} u^h(c(s^t)).
\]

**Budget Constraint.** Define the 1-period payoff vector (in units of numeraire) at node \( s^t \) by

\[
R(s^t) = d(s^t) + q(s^t)b(s^t).
\]
Household h chooses, at each node $s^t \in N^h$ a level of consumption $c^h(s^t)$ and a $k(s^t)$-vector of securities $z^h(s^t)$ to hold at the end of trading, subject to the budget constraints:

$$c^h(s^0) + q(s^0)'z^h(s^0) \leq w^h(s^0) + q(s^0)'\tilde{z}^h(s^0),$$

and at each node $s^t \neq s^0$,

$$c^h(s^t) + q(s^t)'z^h(s^t) \leq w^h(s^t) + R(s^t)'z^h(s^t - 1),$$

with

$$c^h() \geq 0$$
$$q(s^t)'z^h(s^t) \geq -B^h(s^t),$$

where $B^h : N \rightarrow R_+$ indicates an exogenous and non-negative household specific borrowing limit at each node. We assume households take the borrowing limits as given, just as they take security prices as given.

**Market Clearing conditions.** At each $s^t$,

$$\sum_{h \in H(s^t)} c^h(s^t) = \bar{w}(s^t)$$
$$\sum_{h \in H(s^t)} z^h(s^t) = z(s^t).$$

**Definition 4** Given the price process $q$, we say that no arbitrage opportunities exist at $s^t$ if there is no $z \in \mathbb{R}^k(s^t)$ such that

$$R(s^{t+1})z \geq 0, \text{ for all } s^{t+1}|s^t,$$
$$q(s^t)z \leq 0,$$

with at least one strict inequality.

**Lemma 5** When $q$ satisfies the no-arbitrage condition at $s^t$, there exists a set of state prices (a SDF) $a(s^t) > 0$ and $\{a(s^{t+1})\}$ with $a(s^{t+1}) > 0$ for all $s^{t+1}|s^t$, such that the vector of asset prices at $s^t$ can be written as

$$q(s^t) = \frac{1}{a(s^t)} \sum_{s^{t+1}|s^t} a(s^{t+1})R(s^{t+1}).$$

**Proof.** As usual, proof follows from applying the separation result.

Note that if for a given price process $q$, there are no arbitrage opportunities at any $s^t$ individually, then we can apply the lemma repeatedly and define some state-price process $\{a(s^t)\}$ for the entire tree, for which the pricing equation holds. Let $A(s^t)$ denote the set of such processes for the subtree with root $s^t$. Only under complete markets is the set $A(s^t)$ a singleton.

As a remark, note that completeness is an endogenous property since one-period payoffs $R$ contain asset prices. Therefore, completeness cannot be assessed ex ante but only at each given equilibrium.
Definition 6. For any state-price process \( a \in A(s^t) \), define the \( k(s^t) \)-vector of fundamental values for the securities traded at node \( s^t \) by

\[
f(s^t, a) = \frac{1}{a(s^t)} \sum_{T=t+1}^{\infty} \sum_{s^T \mid s^t} a(s^T) x(s^T | s^t).
\] (10)

Observe that the fundamental value of a security is defined with reference to a particular state-price process, however the following properties it displays are true regardless of the state prices chosen.

Proposition 7. At each \( s^t \in N \), \( f(s^t, a) \) is well-defined for any \( a \in A(s^t) \) and satisfies

\[
0 \leq f(s^t, a) \leq q(s^t).
\]

Proof. From equation (9), we have

\[
a(s^t)q(s^t)' = \sum_{s^{t+1} \mid s^t} a(s^{t+1}) x(s^{t+1} | s^t) + \sum_{s^{t+1} \mid s^t} a(s^{t+1}) q(s^{t+1})' e(s^{t+1} | s^t)
\]

and, iterating on this equation we obtain

\[
a(s^t)q(s^t)' = \sum_{T=t+1}^{T^t} \sum_{s^T \mid s^t} a(s^T) x(s^T | s^t) + \sum_{s^T \mid s^t} a(s^T) q(s^T)' e(s^T | s^t)
\]

for any \( T^t > t \). Since \( e(s^T | s^t) \geq 0 \) by construction, \( q(s^T) \) is non-negative by definition of equilibrium (in the paper) and \( a \in A(s^t) \) is a positive state-price vector, the second term on the RHS is non-negative. So,

\[
a(s^t)q(s^t)' \geq \sum_{T=t+1}^{T^t} \sum_{s^T \mid s^t} a(s^T) x(s^T | s^t).
\]

Note that the RHS is a nondecreasing series in \( T^t \) that is bounded above and, therefore, must converge. So,

\[
a(s^t)q(s^t)' \geq a(s^t)f(s^t, a)
\]

and since \( a(s^t) > 0 \), we get the desired result. \( \blacksquare \)

We can correspondingly define the vector of asset pricing bubbles as

\[
\sigma(s^t, a) = q(s^t) - f(s^t, a),
\] (11)

for any \( a \in A(s^t) \) for the \( k(s^t) \) securities traded at \( s^t \). It follows from the proposition that

\[
0 \leq \sigma(s^t, a) \leq q(s^t),
\]
for any \( a \in A(s^t) \). This corollary is known as the “impossibility of negative bubbles” result. Substituting (11) and (10) into (9) yields

\[
\sigma(s^t) = \frac{1}{a(s^t)} \sum_{s^{t+1}|s^t} a(s^{t+1}) \sigma(s^{t+1}) e(s^{t+1}|s^t).
\]

This is known as the ‘martingale property’ of bubbles.

Quoting from the paper, two important remarks arise from this formula. First, if there exists a (nonzero) price bubble on any security at date \( t \), there must exist a bubble as well on some securities at date \( T \), with positive probability, at every date \( T \geq t \). Second, if there exists a bubble on any security in positive net supply, then there must have existed a bubble as well on some security in positive net supply at every predecessor of the node \( s^t \).

The proposition also offers a corollary that deals with finite-maturity assets. Even with incomplete markets, we have that

\[
f_j(s^t, a) = f_j(s^t),
\]

and

\[
q_j(s^t) = f_j(s^t),
\]

so that there are no pricing bubbles for securities with finite maturity. It is remarkable that we reached this conclusion just by no-arbitrage.

In the case of securities of infinite maturity in an economy with incomplete markets, the fundamental value need not be the same for all state-price processes consistent with the available securities returns. But even in this case, we can define the range of variation in the fundamental value, given the restrictions imposed by no-arbitrage.

### 3.1 Bounds for fundamental values.

**Definition 8** (present value of a stream of dividends) Let \( x : N \rightarrow R_+ \) denote a stream of non-negative dividends. For any \( s^t \), pick any \( a \in A(s^t) \). Then we define the present value at \( s^t \) of \( x \) with respect to \( a \) by

\[
V_x(s^t, a) = \frac{1}{a(s^t)} \sum_{T= t+1}^{\infty} \sum_{s^T|s^t} a(s^T) x(s^T).
\]

Since this present value depends on the SDF picked, let us now define the bounds for the present value at \( s^t \) of dividends \( x \).

**Definition 9** For any \( s^t \), suppose \( \{x(s^r)\} \) has \( x(s^r) \geq 0, \forall s^r|s^t, r > t \). Define

\[
\pi_x(s^t) = \inf_{a \in A(s^t)} \{V_x(s^t, a)\}
\]

\[
\pi_x(s^t) = \sup_{a \in A(s^t)} \{V_x(s^t, a)\}.
\]

A few remarks follow from these definitions. First note that these definitions are conditional on a given price process \( q \) since the set of no-arbitrage SDFs are defined with respect to \( q \). Next, observe that for any security \( x^j \), \( \pi_x^j(s^t) \leq f^j(s^t, a) \leq \pi_x^j(s^t) \), for all \( a \in A(s^t) \). Finally, note that \( \pi_x^j(s^t) < q^j \) implies that there is a pricing bubble for security \( j \).
3.2 Borrowing Limits

Recall that to rule out Ponzi schemes when agents are infinitely lived, a lower bound on individual wealth is needed. Let us define a particular type of borrowing limit.

**Definition 10** We shall say that an agent’s borrowing ability is only limited by her ability to repay out of her own future endowment if

\[ B^h(s^t) = \pi_w(s^t), \tag{12} \]

for each \( s^t \in N^h \setminus N^h \).

It can be shown that these borrowing limits never bind at any finite date. They are just a constraint on the asymptotic behavior of a household’s debt. These borrowing limits are a generalization, to the case of incomplete markets, of the familiar “present-value budget constraint” for the complete markets case. In the case of finitely lived agents, these borrowing limits are equivalent to imposing no borrowing limits at all nonterminal nodes.

An important consequence of this specification is the following.

**Proposition 11** Suppose that household \( h \) has borrowing limits of the form (12). Then the existence of a solution to the agent’s problem for given prices \( q \) implies that \( \pi_w(s^t) < \infty \), at each \( s^t \in N^h \), so that there is a finite borrowing limit at each node.

Note that if more stringent borrowing limits were imposed, i.e. \( B^h(s^t) < \pi_w(s^t) \), we can have equilibria where \( \pi_w(s^t) = \infty \). Recall this comment in the Bewley’80 example.

3.3 Bubbles when aggregate wealth is finite

**Proposition 12** (prop 2.5) Consider an equilibrium \( \{q, \pi^h, z^h\} \). Suppose that the (maximum) value of aggregate wealth is finite, i.e. \( \pi_w(s^t) < \infty \). Suppose also that there exists a bubble on some security in positive net supply at \( s^t \) so that \( \sigma(s^t)'z(s^t) > 0 \). Then, \( \forall K > 0 \), there exists a time \( T \) and \( s^T | s^t \) such that

\[ \sigma(s^T)'z(s^T) > K \tilde{w}'(s^t). \]

That is, there is a positive probability that the total size of the bubble on the securities becomes an arbitrarily large multiple of the value of the aggregate supply of goods in the economy. The proof exploits crucially the martingale property of bubbles.

It follows from this result that some agent must accumulate vast wealth at the same time that the value of her consumption is going to zero.
3.4 Main result

We already learned that no bubbles can arise in securities with finite maturity. The next theorem extends the result to securities in positive net supply as long as we are at equilibria with finite aggregate wealth. The proof uses the nonoptimality of the behavior implied by the previous proposition.

**Theorem 13 (theo 3.3)** Let \( \{ q, c^h, \pi^h \} \) be an equilibrium. Suppose that at each node \( s^t \in N \), there exists \( a \in A(s^t) \) such that \( V_w(s^t, a) < \infty \). Then

\[ q^j(s^T) = f^j(s^T, a), \]

for all \( s^T | s^t \) and \( a \in A(s^t) \), for each security \( j \) traded at \( s^T \) that has finite maturity or positive net supply.

Note that if we have that at equilibrium \( \pi_w(s^t) < \infty \), the condition of the theorem is satisfied.

3.5 Sufficient Conditions

The next two corollaries to the theorem provide conditions on the primitives of the model that guarantee that the value of aggregate wealth is finite at any equilibrium.

**Corollary 14 (cor. 3.4)** Suppose that there exists a portfolio \( \hat{z} \in R_k(s^0) \) such that

\[ x(s^t | s^0) \hat{z} \geq w(s^t), \quad \forall s^t \in N. \]

Then theorem 3.3 applies at any equilibrium.

Intuitively, if the existing securities allow such a portfolio to be formed, it must have a finite price at any equilibrium. But since the dividends paid by this portfolio are higher at every state than the aggregate endowment, the equilibrium value of the aggregate endowment is bounded by a finite number.

**Corollary 15 (cor. 3.5)** Suppose that there exists an (infinitely lived) agent and \( \varepsilon > 0 \) such that i) \( w^h(s^t) \geq \varepsilon w(s^t), \quad \forall s^t \in N \) and ii) \( B^h(s^t) = \pi_w^h(s^t), \quad \forall s^t \in N \) and for all \( h \). Then theorem 3.3 applies at any equilibrium.

The intuition for this result is as follows. The borrowing limits imposed allow all agents (and in particular the infinitely lived one) to “issue” an IOU against his or her stream of endowments. Again, in equilibrium, the asset issued by the infinitely lived agent must have a finite price. This implies that a fraction of aggregate wealth has a finite value in equilibrium, which obviously implies that aggregate wealth is finite.

As a remark, note that these two corollaries share the same spirit. They both state that by adding some assets to the economy, we can rule out bubbles in equilibrium (for securities in positive net supply). Therefore, the conclusion of the paper is that bubbles are a non-robust phenomenon (in securities in positive net supply).
3.6 (Famous) Examples of Bubbles

Recall that fiat money is a security that pays no dividends. Its only return comes from paying one unit of itself in the next period. Therefore if fiat money is in net supply and has a positive price in equilibrium, that is a bubble. The following two models have equilibria with such a property. Theorem 3.3 implies that in those equilibria aggregate wealth must have an infinite value. See the paper for the primitives of each model.

1. Samuelson (1958)’s OLG model. Observe how all the assumptions of corollary 3.5 hold except the existence of an agent that owns a fraction of aggregate wealth.

2. Bewley (1980). Observe that now the only assumption of corollary 3.5 that fails is that the borrowing constraints are more stringent than those required in the corollary.

Problem set 3. Read the primitives of the model from the paper and show that each model has equilibria where fiat money has a positive value. (As an extra question, what happens to these equilibria if we add a 1-period bond on top of the fiat money?)

4 Asset Pricing with Trading Constraints

When agents are heterogenous in endowments, the existence of trading restrictions (e.g. no borrowing) makes that, in equilibrium, some agents will be effectively constrained by the restriction, whereas some others will not be. This implies that when such restrictions are present, it is not true that any agent’s MRS is a SDF (i.e. can be used to price assets). This poses two questions. First, does no-arbitrage retain its pricing implications? And second, is there any relation between the set of individual equilibrium MRSs and the set of SDFs or, in other words, what are the HJ bounds when trading restrictions are present?

The following two papers address these questions within the class of two-period economies with possibly incomplete markets.

4.1 Luttmer (Ecta, 1996)

The main contribution of this paper is the derivation of the asset pricing implications of No Arbitrage when there exists restrictions of the set of portfolios that can be formed. The restrictions are supposed to apply to all agents in the economy. An example of such restrictions is when agents can only buy assets but not sell them.

Consider a two-period economy with $J$ assets, $S$ periods and asset matrix $A$. Define the set of attainable payoffs to be

$$M = \{ x \in R^S : x = \theta'A, \theta \in C \},$$

where $C$ is a cone (which is a nice convex set). If $C = R^J$ then we have no restrictions on trading and we are back to the standard model.
Definition 16 \((M, q)\) are No Arbitrage if and only if
\[
x \in M \cap R^S_{++} \implies q(x) > 0.
\]

The implications of no arbitrage for asset pricing are given in the next version of the No-Arbitrage theorem for economies with trading restrictions.

Theorem 17 Suppose that \((M, q)\) are No Arbitrage. Then there exists a strictly positive SDF, \(y \in R^S_{++}\) such that
\[
q - E(y'A) \in C^*, \text{ where } C^* = \{v \in R^J : v'\theta \geq 0, \theta \in C\}.
\]

It is worth noting that the (omitted) proof is based on the same separation result as the standard no-arbitrage theorem.

To illustrate what the set \(C^*\) means, consider two examples. If \(C = R^J\) then \(C^* = \{0\}\), and the theorem reduces to our familiar no-arbitrage theorem.

In the case where agents are restricted to only buying assets, \(C = R^J_+\), which implies \(C^* = R^J_+\), i.e. \(q - E(y'A) \geq 0\). Recall from problem set 1 the Kuhn-Tucker conditions for optimal asset holdings of agents in equilibrium. Those conditions provide intuition for the pricing inequality. When in equilibrium an agent would love to sell an asset but is not allowed to do so, we obtained that \(q^j - E(MRS^jA) > 0\). And, for those agents for which the trading restrictions aren’t binding in equilibrium, we obtain the familiar equality.

**HJ bounds.** When the the trading restrictions are assumed to be “only buying” assets, we get the following modification in the HJ bounds.

Corollary 18 Suppose \(C = R^J_+\). Then by the no-arbitrage theorem, there exists \(y \in R^S_{++}\) such that
\[
q \geq E(y'A), \text{ and } m^i \leq y, \text{ for all } i,
\]

where \(m^i\) is the individual MRS for agent \(i\).

More plainly, the corollary states that as a consequence of no-arbitrage, there is a strictly positive SDF that bounds above all individual MRSs.

The fact that the no-arbitrage theorem under trading restrictions loosens the pricing implications (roughly, we get an inequality where we used to get an equality) implies much larger HJ bounds. This can, in principle, explain the equity premium puzzle but, on the other hand, makes the set of admissible SDFs to be “too large” for the HF bounds to be a useful test of the asset pricing theory.

4.2 He and Modest (JPE, 1995)

This paper looks at the asset pricing implications for economies where agents have a specific trading constraint, agents can never have negative wealth. This
is the (already familiar) restriction that each agent is only allowed to borrow up to the present value of its stream of endowments (recall Santos and Woodford).

Recall that no arbitrage implies that returns (asset payoff vector divided by its price) have unit price and that excess returns (the difference of two returns) have price zero. Therefore, the assumed trading constraint places no constraint at all on the trading of excess returns.

**Result 1.** It follows that despite the trading constraint, we can still use any agent’s MRS to price excess returns. That is, if \( y \) is any SDF, and \( a \) is any excess return,

\[
E(m^i a^i) = E(y a^i),
\]

where \( m^i \) is the MRS of any agent.

**Result 2.** An implication of the previous result is that there is a SDF with the property that all individual MRSs are proportional to it, and bounded above by it i.e. there exists \( y \in R_{++}^S \) such that

\[
m^i = \Psi^i y, \quad \Psi^i \leq 1, \text{ for all } i.
\]

Intuitively, the proportionality of the MRSs follows from unrestricted trading of excess returns, whereas the upper bound on MRSs is the Luttmer result.3

**question:** are the trading restrictions assumed in this paper a particular case of those in Luttmer (the cone)?

## 5 When do Incomplete Markets matter?

Let’s go back to economies with no trading restrictions and infinitely lived agents. Of course, we still need to impose some borrowing limit to rule out Ponzi schemes but we know how to do that in a non-binding way, as we learned from Santos and Woodford.

### 5.1 The puzzle: “Empirically, Incomplete Markets does not matter”

In a series of papers, Telmer (JF, 1993), Aiyagari (QJE, 1994) and Krusell and Smith (JPE, 1996) among others, different authors have found support for a puzzling result. Even though theoretically the effects of complete or incomplete markets on equilibrium allocations and prices are crucial, empirically, they do not seem to matter significantly. These are all infinite-horizon economies with agents facing idiosyncratic risk due to stochastic endowments and a seriously incomplete asset structure.

The next two papers illuminate the question from the theoretical viewpoint. The current state of the literature is that the stationarity or not of the individual endowment process is key to the effects of incomplete markets. This is the result

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3For a good reference on these issues, see LeRoy and Wenner, chapters 4 and 7. In particular, they cover the equilibrium pricing implications in models with borrowing constraints and bid-ask spreads.
from Levine and Zame (2001). A second contribution is Constantinides and Duffie (JPE, 1996), who show that with incomplete markets and nonstationary endowment processes (i.e. permanent shocks), “anything is possible”.

5.2 Levine and Zame (2001)

For short horizons, we know that market incompleteness generally matters because of agents’ inability to insure against bad shocks. However, for long horizons, market incompleteness may not matter if traders can self insure i.e. if they can borrow in bad times and save in good times.

5.2.1 The Model

**Time and Uncertainty.** Let $S$ denote the tree. Let $s \leq s'$ mean that state $s'$ is an immediate successor of state $s$. For any node $s$, let $s^-$ denote its (unique) predecessor, $s^+$ denote the set of its immediate successors, and let $t(s)$ be the length of the history from initial node $0 \in S$ up to $s$.

Exogenous events are supposed to follow a finite Markov chain with state space $\Omega$ and strictly positive transition probabilities. For any $s$ and $\sigma \in s^+$, define the conditional probability of each successor of $s$ by $\pi(\sigma|s)$ from the transition probabilities of the Markov chain. For each $s$, let $\pi_s$ denote the unconditional probability form the root node to node $s$.

**Commodities.** We will simplify their model by imposing $L=1$ from the beginning. Let the consumption set be $X = l^\infty(S)_+$ i.e. the set of bounded sequences. A consumption plan is thus a non-negative bounded sequence, or equivalently, a mapping $x : S \rightarrow \mathbb{R}_+^\infty$.

**Securities.** Suppose that $J$ securities are available at each $s$. These securities are short-lived (actually, 1-period lived) and pay in units of goods. Security $j$ traded at $s$ yields $A_j(\sigma)$ units of good at $\sigma \in s^+$. Let $\theta \in \mathbb{R}^J$ be a portfolio traded at $s$ then define the dividends of this portfolio at the successor nodes to $s$ to be $\text{div}_s \theta = \sum_{j=1}^J \theta_j A_j(\sigma)$, for $\sigma \in s^+$. Let the security prices be a mapping $q : S \rightarrow \mathbb{R}^J$.

**Utility.** The economy is populated by $N$ infinitely lived agents with utility $U^i : l^\infty(S)_+ \rightarrow \mathbb{R}$,$ U^i = (1-\delta) \sum_{t=0}^\infty \delta^t \sum_{t(s)=t} \pi_s u^i(x_s)$, where $u^i$ is $C^2$, strictly increasing and strictly concave.

**Endowments.** Agent $i$’s endowment process is a mapping $e^i : S \rightarrow \mathbb{R}_+$ with the restrictions that i) it only depends on the current (Markov) state, i.e. $e^i(s) = e^i(\omega_s)$, where $\omega_s \in \Omega$, and ii) for each agent and state, $e^i(s)$ is bounded away from zero.

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4 Actually, we just need the chain to be recurrent.
Budget sets and debt constraints. Taking asset prices $q$ as given, agent $i$ chooses a consumption plan $x^i : S \rightarrow R_+$ and a portfolio plan $\theta^i : S \rightarrow R^J$ subject to the following spot budget constraint: for each $s \in S$,

$$x^i_s + q_s \theta^i_s = e^i_s + \text{div}_s \theta^i_s.$$  \hspace{1em} (13)

Note that if $\theta^i_s$ is negative it means that the agent is selling asset $j$, i.e. the agent is borrowing. Of course, since agents are infinitely lived, we need a No-Ponzi game condition (i.e. a limit on borrowing). The next couple of definitions build up toward defining borrowing limits that allow the agent to trade any portfolio plan that implies a debt that can be repaid in finite time. First, we’ll define debt and then we’ll define what it means to repay in finite time.

Let $\theta$ be a portfolio plan, define debt at node $s$ as

$$d_s \equiv -\text{div}_s \theta_s^-.$$

If this quantity is positive, an agent following portfolio plan $\theta$ is in debt when entering date-event $s$. To meet this debt, the agent must raise income from the sale of endowment and/or selling securities (i.e. borrowing).

Let us define next an upper bound on the level of debt that agent $i$ can hold at node $s$, $d_s$. We will say that debt $d_s \geq 0$ can be repaid in finite time from $s$ if there are consumption and portfolio plans, $y$ and $\phi$, and a time $T$ such that

i. at $s$, $y_s + q_s \phi_s + d_s \leq e^i_s$

ii. $y_{s+}, \phi_{s+}$ satisfy the spot budget constraint (13) at every $\sigma \geq s$

iii. if $\sigma \geq s$ and $t(\sigma) - t(s) \geq T$ then $d_\sigma \leq 0$.

That is, the plans meet the liability $d_s$ at node $s$, meet the spot budget constraints at every successor node and leave no debt at any node following $s$ by $T$ or more periods.

Finally, define the debt constraint as

$$D^i_s = \sup \{ d : d \text{ can be repaid in finite time from } s \}.$$

The budget set for agent $i$ at prices $q$ is then

$$B^i(q, D^i) = \left\{ (x^i, \theta^i) : \begin{array}{l}
\text{for each } s,
\quad x^i_s + q_s \theta^i_s = e^i_s + \text{div}_s \theta^i_s,
\quad d_s = -\text{div}_s \theta^i_s \leq D^i_s, \text{ for each } \sigma \in s^+ \end{array} \right\}.$$

Note that the trades at date-event $s$ are constrained by limiting the debt that can be carried to each of the date-events that immediately follow $s$.

Equilibrium. $\{q, (x^i), (\theta^i)\}$ is an equilibrium if

i. markets clear at each $s$,

$$\sum_{i=1}^{N} (x^i_s - e^i_s) = 0$$

$$\sum_{i=1}^{N} \theta^i_s = 0$$

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ii. and agents maximize utility,
\[(x^i, \theta^i) \in B^i(q) \]
\[(y^i, \phi^i) \in B^i(q) \implies U^i(x^i) \geq U^i(y^i).\]

Levine and Zame (1996) guarantees the existence of equilibrium. However, they note that Markov equilibria need not exist.

5.2.2 Results when only Idiosyncratic Risk

Extra assumptions. Recall we already assumed stationary of endowment processes (in particular, a non-recurrent Markov chain) and one consumption good (L=1). In addition, consider the following assumptions.

A1. No aggregate risk: the social endowment \( e = \sum_{i=1}^N e^i \) is independent of \( s \).

A2. Precautionary savings: for each \( i \), \( Du^i \) is (weakly) convex.

A3. At each \( s \), a riskless bond, yielding one unit of consumption at each \( \sigma \in s^+ \), is available for trade.

For a given \( \delta \), let \( EQ_\delta \) denote the set of equilibria in the economy. Let \( \zeta \in EQ_\delta, \zeta = (q, (x^i, \theta^i)) \) be denote any equilibrium.

The previous extra assumptions imply the following two remarks.

Remark 1: (permanent income) the stationarity of the endowment process implies that for each individual, we can calculate the long-run average endowment, denoted by \( e^i \) for agent \( i \).

Remark 2: (Pareto optimal allocations) For any given \( \delta \), the efficient allocations, \( PO_\delta \), are given by the \( N \)-tuples of fixed shares of the constant social endowment. And, in particular, the perfect risk-sharing allocation \( \bar{\pi} = (\bar{\pi}_1, ..., \bar{\pi}_N) \), at which each agent consumes his permanent income, is Pareto optimal.

Theorem 19 Under the extra assumptions, when \( \delta \) is sufficiently close to 1, every equilibrium is close to perfect risk sharing, in the sense that i) equilibrium utilities are close to the utilities of the perfect risk sharing allocation, ii) the time-discounted probability that equilibrium consumptions deviate from the perfect risk sharing allocation by more than a given amount is small, and iii) the time-discounted probability that equilibrium asset prices deviate from risk neutral pricing by more than a given amount is small.

The proof provides a lower bound on equilibrium utility by constructing a budget feasible plan whose utility is almost that of constant average consumption. A crucial step in the argument is establishing that the riskless interest rate is bounded above, with a bound close to zero. This is important because the budget feasible plan they construct is financed by borrowing, and a low interest rate makes borrowing easy.

5.2.3 Aggregate Risk

In general, in the presence of aggregate risk, market incompleteness matters even if endowment processes are stationary (i.e. shocks are transitory). The
reason is the following. When there is aggregate risk, the upper bound on the interest rate need not obtain; when the aggregate endowment is low, many traders will want to borrow, and this demand for loans may drive up the riskless interest rate. A high interest rate interferes with risk sharing because it makes borrowing difficult. Summing up, aggregate risk matters because it affects asset prices.

If we assume that the aggregate risk is tradeable (i.e. that there exist the right assets) then we can again get the result of the previous theorem.

5.2.4 Two goods

When there is more than one consumption good, market incompleteness matters again, even without aggregate risk. The reason is that commodity prices provide another source of untraded risk.

5.2.5 Conclusion

In a one-good economy populated by infinitely-lived, patient agents, market incompleteness will not matter if shocks are transitory and risk is purely idiosyncratic. When there is aggregate uncertainty or more than one consumption good, market incompleteness matters, in general.

The next paper adds to the question by showing that if shocks are permanent, market incompleteness matters too.

5.3 Constantinides and Duffie (JPE, 1996)

5.4 Empirically, are endowments stationary?

Exploiting the PSID database, a number of papers have tried to determine whether people’s endowment processes are stationary or not. Heaton and Lucas (JPE, 1996) find evidence of nonstationarity whereas Cochrane finds evidence in the opposite direction.

6 Endogenous Incomplete Markets