Appendix C

C.1 Proof of Proposition 9 in Appendix A.3

Note first that

\[ q(k, \phi, B + dB) = \max_i \mathbb{E} \text{MRS}_i^d(s) \left[ f(k, \phi; s) - B - dB \right]. \]

Since for all \( i \notin I^e \), \( \mathbb{E} \text{MRS}_i^d(s) \left[ f(k, \phi; s) - B \right] < q(k, \phi, B) \), the max in the above expression is attained for some \( i \in I^e \) and hence

\[ q(k, \phi, B + dB) = q(k, \phi, B) + \max_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s) \left[ dB \right]. \]

The right and left derivative of \( q(k, \phi, B) \) with respect to \( B \) are then given by:

\[ \frac{\partial q}{\partial B_+} = -\min_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s); \quad \frac{\partial q}{\partial B_-} = -\max_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s) \]

and may differ. Similarly the derivatives with respect to \( k \) are:

\[ \frac{\partial q}{\partial k_+} = \max_{i \in I^e} \mathbb{E} \left[ \text{MRS}_i^d(s) f_k(s) \right]; \quad \frac{\partial q}{\partial k_-} = \min_{i \in I^e} \mathbb{E} \left[ \text{MRS}_i^d(s) f_k(s) \right] \]

where \( f_k \) denotes the derivative of \( f \) with respect to \( k \).

The first order conditions are then different according to whether the no default constraint (2) binds or not. Recalling that \( s \) denotes the lowest output state they are given by:

i. \( f(k, \phi; s) > B \) and

\[ \frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p \leq 0, \quad \frac{\partial V}{\partial k_+} = \frac{\partial q}{\partial k_+} - 1 \leq 0, \]
\[ \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p \geq 0, \quad \frac{\partial V}{\partial k_-} = \frac{\partial q}{\partial k_-} - 1 \geq 0; \]

Since (44) implies that \( \frac{\partial q}{\partial B_+} \geq \frac{\partial q}{\partial B_-} \), the above conditions (with respect to \( B \)) are equivalent to:

\[ \frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p = \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p = 0, \]

that is:

\[ \max_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s) = \min_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s) = p = \max_{i \in I^e} \mathbb{E} \text{MRS}_i^d(s) \]
or (35) holds. Similarly, from (45) we see that \( \frac{\partial q}{\partial k_+} \geq \frac{\partial q}{\partial k_-} \), the above conditions (with respect to \( k \)) are equivalent to:

\[
\frac{\partial q}{\partial k_-} - 1 = \frac{\partial q}{\partial k_+} - 1 = 0,
\]

that is,

\[
\max_{i \in I^e} \mathbb{E} \left[ MRS^i(s) f_k(s) \right] = \min_{i \in I^e} \mathbb{E} \left[ MRS^i(s) f_k(s) \right] = 1
\]
or (36) holds.

ii. \( f(k; \phi; s) = B \) and

\[
\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p \geq 0, \quad \frac{\partial V}{\partial k_-} = \frac{\partial q}{\partial k_-} - 1 \leq 0.
\]

This condition can be equivalently written as

\[
p = \max_i \mathbb{E} MRS^i(s) \geq \max_{i \in I^e} \mathbb{E} MRS^i(s)
\]

and

\[
1 \geq \max_{i \in I^e} \mathbb{E} \left[ MRS^i(s) f_k(s) \right] .
\]

Note that (48) is always satisfied. In particular, it holds as equality when at least one equity holder is also a bond holder, or \( I^e \cap I^d = \emptyset \), and as a strict inequality when no equity holder is also a bond holder, or all equity holders would like to short the risk free asset.

To verify whether a solution indeed obtains at \( f(k; \phi; s) = B \) when (49) holds, we need to consider also the optimality with respect to joint changes\(^{50} \) in \( k \) and \( B \) or\(^{51} \):

\[
\begin{align*}
\frac{\partial V}{\partial B_+} dB + \frac{\partial V}{\partial k_+} dk &= \left( \frac{\partial q}{\partial B_+} + p \right) dB + \left( \frac{\partial q}{\partial k_+} - 1 \right) dk \leq 0 \text{ for } dB = f_k(s) dk > 0 \\
\frac{\partial V}{\partial B_-} dB + \frac{\partial V}{\partial k_-} dk &= \left( \frac{\partial q}{\partial B_-} + p \right) dB + \left( \frac{\partial q}{\partial k_-} - 1 \right) dk \geq 0 \text{ for } dB = f_k(s) dk < 0
\end{align*}
\]

Using again (44),(45) to substitute for the derivatives of \( q \) w.r.t. \( B \) and \( k \) into the first of the above expressions yields:

\[
\begin{align*}
&f_k(s) \left( - \min_{i \in I^e} \mathbb{E} MRS^i(s) + \max_i \mathbb{E} MRS^i(s) \right) + \\
&\max_{i \in I^e} \mathbb{E} \left( MRS^i(s) f_k(s) \right) - 1 \leq 0,
\end{align*}
\]

\(^{50}\)This is obviously not necessary when the first order conditions are satisfied at an interior solution, that is when (35) and (36) hold.

\(^{51}\)Without loss of generality, we can limit our attention to changes in \( B \) and \( k \) such that the no default constraint still binds, or \( f_k(s) dk \geq dB \) holds as equality.
or

\[
f_k(s) \left( - \min_{i \in I^c} \mathbb{E} MRS^i(s) + \max_{i} \mathbb{E} MRS^i(s) \right) \leq 1 - \max_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right),
\]

where the term on the r.h.s. is always nonnegative by (49) and the one on the l.h.s. is obviously always nonnegative. Proceeding similarly with the second expression above, we get:

\[
\begin{bmatrix}
f_k(s) \left( - \min_{i \in I^c} \mathbb{E} MRS^i(s) + \max_{i} \mathbb{E} MRS^i(s) \right) \\
+ \min_{i \in I^c} E_{s_0} \left( MRS^i(s_1) f_k(s) \right) - 1
\end{bmatrix} \geq 0,
\]
or

\[
1 - \min_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right) \leq f_k(s) \left( - \min_{i \in I^c} \mathbb{E} MRS^i(s) + \max_{i} \mathbb{E} MRS^i(s) \right),
\]

and again both terms are nonnegative.

Putting (50) and (51) together yields

\[
1 - \max_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right) \geq f_k(s) \left( - \min_{i \in I^c} \mathbb{E} MRS^i(s) + \max_{i} \mathbb{E} MRS^i(s) \right) \geq 1 - \min_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right),
\]

where the second inequality follows from the fact that

\[- \min_{i \in I^c} \mathbb{E} MRS^i(s) \geq - \max_{i \in I^c} \mathbb{E} MRS^i(s) \]

Since, by the same argument,

\[- \min_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right) \geq - \max_{i \in I^c} \mathbb{E} \left( MRS^i(s) f_k(s) \right),
\]

50
the above condition can only hold as equality:

\[
1 - \max_{i \in I^c} \mathbb{E} \left( MRS_i^i(s) f_k(s) \right) = \]

\[
f_k(s) \left( - \min_{i \in I^c} \mathbb{E} MRS_i^i(s) + \max_i \mathbb{E} MRS_i^i(s) \right) = \]

\[
f_k(s) \left( - \max_{i \in I^c} \mathbb{E} MRS_i^i(s) + \max_i \mathbb{E} MRS_i^i(s) \right) = \]

\[
= 1 - \min_{i \in I^c} \mathbb{E} \left( MRS_i^i(s) f_k(s) \right) \]

This implies that (37), (38), (39) hold, thus completing the proof.■

C.2 Characterization of the firms’ optimality conditions with risky debt

We study here in detail the economy where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms’ decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to \( B \), let \( S^{nd} \) denote the collection of states in \( t = 1 \) for which \( f(k; s) \geq B \) and by \( s^{nd} \) the lowest state in \( S^{nd} \), that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote \( S^d \) the collection of states in \( t = 1 \) for which \( f(k; s) < B \), i.e. the firm (partially) defaults on its debt.

**Proposition 10**  The optimal investment and capital structure decision of a firm obtains either at an interior solution, where \( f(k; s^{nd}) > B \), with:

\[
p = \min_{i \in I^d} \mathbb{E} \left( MRS_i(s) \left[ \frac{f(k; s)}{B} \right] \mid s \in S^d \right) \text{Pr}\{s \in S^d\} + \]

\[
\min_{i \in I^d} \mathbb{E}(MRS_i(s) \mid s \in S^{nd}) \text{Pr}\{s \in S^{nd}\} = \]

\[
= \max_{i \in I^d} \mathbb{E} \left( MRS_i(s) \left[ \frac{f(k; s)}{B} \right] \mid s \in S^d \right) \text{Pr}\{s \in S^d\} + \]

\[
\max_{i \in I^d} \mathbb{E}(MRS_i(s) \mid s \in S^{nd}) \text{Pr}\{s \in S^{nd}\}.
\]
and

$$1 = \max_{i \in I_e} \mathbb{E} \left\{ MRS^i(s) f_k(k, s) \mid s \in S^{nd} \right\} \Pr\{s \in S^{nd}\} +$$

$$\max_{i \in I^d} \mathbb{E} \left\{ MRS^i(s) f_k(k, s) \mid s \in S^d \right\} \Pr\{s \in S^d\}$$

$$= \min_{i \in I^e} \mathbb{E} \left\{ MRS^i(s) f_k(k, s) \mid s \in S^{nd} \right\} \Pr\{s \in S^{nd}\} +$$

$$\min_{i \in I^d} \mathbb{E} \left\{ MRS^i(s) f_k(k, s) \mid s \in S^d \right\} \Pr\{s \in S^d\}$$

or at a corner solution, $$f(k; _{x}^{s^{nd}}) = B$$.

Before proving Proposition 10 we state the following Claim, which characterizes the conditions for corner solutions.

Claim 1 The conditions for an optimum at a corner, $$f(k; _{x}^{s^{nd}}) = B$$, are:

$$\min_{i \in I^e} E_{s_0} \left\{ \overline{MRS}^i(s_1) s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\min_{i \in I^d} E_{s_0} (MRS^i(s_1) \left[ f(k; s_1) \right] B) s_1 \in S^{dr} \Pr\{s_1 \in S^{dr}\} \geq p \geq$$

$$\geq \max_{i \in I^e} E_{s_0} \left\{ MRS^i(s_1) s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\max_{i \in I^d} E_{s_0} (MRS^i(s_1) \left[ f(k; s_1) \right] B) s_1 \in S^{d} \Pr\{s_1 \in S^{d}\}$$

$$\min_{i \in I^e} E_{s_0} \left\{ MRS^i(s_1) f_k(k, s_1) s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\min_{i \in I^d} E_{s_0} (MRS^i(s_1) \left[ f_k(k, s_1) \right] B) s_1 \in S^{dr} \Pr\{s_1 \in S^{dr}\} \geq 1 \geq$$

$$\geq \max_{i \in I^e} E_{s_0} \left\{ MRS^i(s_1) f_k(k, s_1) s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\max_{i \in I^d} E_{s_0} (MRS^i(s_1) \left[ f_k(k, s_1) \right] B) s_1 \in S^{d} \Pr\{s_1 \in S^{d}\}$$
1 - \max_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \left\{ f_k(k, s_1) \right\} \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} -

\max_{i \in I^d} E_{s_0} \left( MRS^i(s_1) \frac{f_k(k; s_1)}{B} \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} = 

\left[ - \min_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} 

\left[ - \min_{i \in I^d} E_{s_0} \left( MRS^i(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(s_1^{nd}) = 

\left[ - \min_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} 

\left[ - \max_{i \in I^d} E_{s_0} \left( MRS^i(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(s_1^{nd}) = 

1 - \min_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} 

\left[ - \min_{i \in I^d} E_{s_0} \left( MRS^i(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} \right]

**Proof of Proposition 10** The equity price map in the presence of risky debt is given by

\[ q(k, B) = \max_{i} E_{s_0} \left\{ MRS^i(s_1) \left[ f(k; s_1) - B \right] \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \]

The debt price map is

\[ p(k, B) = \max_{i} \left\{ E_{s_0} \left( MRS^i(s_1) \right) s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \right. \]

\[ + E_{s_0} \left( MRS^i(s_1) \left[ \frac{f(k; s_1)}{B} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} \right\} \]

The statement only refers to the interior case: \( f(k; s_1^{nd}) > B \). Here, the partials of the price maps with respect to \( B \) are

\[ \frac{\partial q}{\partial B} = - \min_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \]

\[ \frac{\partial q}{\partial B} = - \max_{i \in I^d} E_{s_0} \left\{ MRS^i(s_1) \right\} s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \]

and

\[ \frac{\partial p}{\partial B} = - \min_{i \in I^d} E_{s_0} \left( MRS^i(s_1) \left[ \frac{f(k; s_1)}{B^2} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} \]

\[ \frac{\partial p}{\partial B} = - \max_{i \in I^d} \left( MRS^i(s_1) \left[ \frac{f(k; s_1)}{B^2} \right] \right) s_1 \in S^d) \Pr\{s_1 \in S^d\} \]
Analogously, the partials with respect to $k$ are

$$
\frac{\partial q}{\partial k_+} = \max_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| f_k(k, s_1) \right|_{s_1 \in S^{nd}} \right\} Pr\{s_1 \in S^{nd}\}
$$

$$
\frac{\partial q}{\partial k_-} = \min_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| f_k(k, s_1) \right|_{s_1 \in S^{nd}} \right\} Pr\{s_1 \in S^{nd}\}
$$

and

$$
\frac{\partial p}{\partial k_+} = \max_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| \frac{f_k(k, s_1)}{B} \right|_{s_1 \in S^d} \right\} Pr\{s_1 \in S^d\}
$$

$$
\frac{\partial p}{\partial k_-} = \min_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| \frac{f_k(k, s_1)}{B} \right|_{s_1 \in S^d} \right\} Pr\{s_1 \in S^d\}
$$

So, if $f(k; s_1^{nd}) > B$, the FOCs with respect to $B$ are:

$$
\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left( \frac{\partial p}{\partial B_+} B + p \right) =
$$

$$
- \min_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\} + \left( - \min_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| \frac{f(k; s_1)}{B} \right|_{s_1 \in S^d} \right\} Pr\{s_1 \in S^d\} + p \right) \leq 0
$$

$$
\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left( \frac{\partial p}{\partial B_-} - B + p \right) =
$$

$$
- \max_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\} + \left( - \max_{i \in I^d} E_{s_0} \left\{ MRS^d_i(s_1) \left| \frac{f(k; s_1)}{B} \right|_{s_1 \in S^d} \right\} Pr\{s_1 \in S^d\} + p \right) \geq 0
$$

which implies

$$
p = \max_i E_{s_0} \left\{ MRS^d_i(s_1) \left| s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\} +
$$

$$
E_{s_0} \left\{ MRS^d_i(s_1) \left| \frac{f(k; s_1)}{B} \right|_{s_1 \in S^d} \right\} Pr\{s_1 \in S^d\}
$$
and (53). On the other hand, the FOCs with respect to $k$ give:

$$\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left( \frac{\partial P}{\partial B} \right) =$$

$$= -1 + \max_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) f_k(k, s_1) \left| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\max_{i \in I^d} E_{s_0} (MR^d(S_1)) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \leq 0$$

$$\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left( \frac{\partial P}{\partial B} \right) =$$

$$= -1 + \min_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) f_k(k, s_1) \left| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$

$$\min_{i \in I^d} E_{s_0} (MR^d(S_1)) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \geq 0$$

which implies (54). This completes the proof of Proposition 10. $\blacksquare$

**Proof of Claim 1.** Note first that, in this case, $f(k; s_1^{nd}) = B$. Denote by $S^{ndr}_{S_1} \subseteq S^{nd}_{S_1}$ the collection of states in $t = 1$ for which the firm does not default, after marginal deviations $dB > 0$ and/or $dk < 0$ (and similarly $S^{dr}_{S_1} \subseteq S^{d}_{S_1}$). Evidently, for marginal deviations $dB > 0$ and/or $dk < 0$ the collection of such states is still given by $S^{nd}_{S_1}$.

The partials of the price maps wrt to $B$ are\(^{52}\)

$$\frac{\partial q}{\partial B_+} = -\min_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) \left| s_1 \in S^{nd}_{S_1} \right\} \Pr\{s_1 \in S^{nd}_{S_1}\}$$

$$\frac{\partial q}{\partial B_-} = -\max_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) \left| s_1 \in S^{nd}_{S_1} \right\} \Pr\{s_1 \in S^{nd}_{S_1}\}$$

and

$$\frac{\partial p}{\partial B_+} = -\min_{i \in I^d} E_{s_0} (MR^d(S_1)) \left[ \frac{f_k(k; s_1)}{B^2} \right] s_1 \in S^d \right\} \Pr\{s_1 \in S^d\}$$

$$\frac{\partial p}{\partial B_-} = -\max_{i \in I^d} E_{s_0} (MR^d(S_1)) \left[ \frac{f_k(k; s_1)}{B^2} \right] s_1 \in S^d \right\} \Pr\{s_1 \in S^d\}$$

Analogously, the partials wrt to $k$ are\(^{53}\)

$$\frac{\partial q}{\partial k_+} = \max_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) f_k(k, s_1) \left| s_1 \in S^{nd}_{S_1} \right\} \Pr\{s_1 \in S^{nd}_{S_1}\}$$

$$\frac{\partial q}{\partial k_-} = \min_{i \in I^e} E_{s_0} \left\{ MR^i(S_1) f_k(k, s_1) \left| s_1 \in S^{nd}_{S_1} \right\} \Pr\{s_1 \in S^{nd}_{S_1}\}$$

\(^{52}\)Obviously, if $S^{nd}_{S_1}$ is a singleton, the right derivative is equal to 0.

\(^{53}\)Obviously, if $S^{nd}_{S_1} = \{s_1\}$ - is a singleton - the left derivative is equal to 0.
and
\[
\frac{\partial p}{\partial k_+} = \max_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^d) \Pr\{s_1 \in S^d\}
\]
\[
\frac{\partial p}{\partial k_-} = \min_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^{d^r}) \Pr\{s_1 \in S^{d^r}\}
\]

So, if \(f(k; s_1^{nd}) = B\), the FOCs wrt \(B\) are:
\[
\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left( \frac{\partial p}{\partial B_+} B + p \right) =
\]
\[
- \min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d(s_1) \left| s_1 \in S^{nd} \right. \right\} \Pr\{s_1 \in S^{nd}\} +
\]
\[
\left( - \min_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f(k; s_1)}{B^2} \right] s_1 \in S^{d^r}) \Pr\{s_1 \in S^{d^r}\} B + p \right) \leq 0
\]
\[
\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left( \frac{\partial p}{\partial B_-} B + p \right) =
\]
\[
- \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d(s_1) \left| s_1 \in S^{nd} \right. \right\} \Pr\{s_1 \in S^{nd}\} +
\]
\[
\left( - \max_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f(k; s_1)}{B^2} \right] s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} B + p \right) \geq 0
\]
which implies (55). Finally, the FOCs wrt \(k\) are:
\[
\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left( \frac{\partial p}{\partial k_+} B \right) =
\]
\[
- 1 + \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d(s_1) \left| f_k(k, s_1) \right. \right\} \Pr\{s_1 \in S^{nd}\} +
\]
\[
\left( \max_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} B \right) \leq 0
\]
\[
\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left( \frac{\partial p}{\partial k_-} B \right) =
\]
\[
- 1 + \min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d(s_1) \left| f_k(k, s_1) \right. \right\} \Pr\{s_1 \in S^{nd}\} +
\]
\[
\left( \min_{i \in I^d} E_{s_0}(\text{MRS}^d(s_1) \left[ \frac{f_k(k; s_1)}{B} \right] s_1 \in S^{d^r}) \Pr\{s_1 \in S^{d^r}\} B \right) \geq 0
\]
which implies (56). Since now expectations in the terms on the two sides of the inequality are taken over different sets, such condition is a little harder to interpret. In particular we can no longer say that all equity holders have the same valuation for the marginal productivity of capital in the no default states. Rather the condition imposes some relationship between the difference among equity holders and bond holders’ valuation for the marginal productivity of capital in the two situations (\(S^d\) and \(S^{d^r}\)).
We also have to check in this case the optimality of \( k, B \) wrt joint deviations of \( B \) and \( k \). As before, without loss of generality, we can restrict our attention to changes of \( B \) and \( k \) such that \( f(k; \bar{z}^{nd}_2) = B \) keeps holding (the set of states for which default occurs does not change).

\[
\frac{\partial V}{\partial B} dB + \frac{\partial V}{\partial k} dk = \left[ \frac{\partial q}{\partial B} + \left( \frac{\partial p}{\partial B} B + p \right) \right] dB + \left[ -1 + \frac{\partial q}{\partial k} + \left( \frac{\partial p}{\partial k} B \right) \right] dk \leq 0,
\]

for \( dB = f_k(\bar{z}^{nd}_1) dk > 0 \); also,

\[
\frac{\partial V}{\partial B} dB + \frac{\partial V}{\partial k} dk = \left[ \frac{\partial q}{\partial B} - \left( \frac{\partial p}{\partial B} B + p \right) \right] dB + \left[ -1 + \frac{\partial q}{\partial k} - \left( \frac{\partial p}{\partial k} B \right) \right] dk \geq 0
\]

for \( dB = f_k(\bar{z}^{nd}_1) dk < 0 \). Substituting the expressions for the partials obtained above, we get

\[
\begin{align*}
&\left[ -\min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d (s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
&- \min_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} + p \right] f_k(\bar{z}^{nd}_1) \\
&- 1 + \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d (s_1) f_k(k; s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \\
&\max_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \leq 0
\end{align*}
\]

or

\[
1 - \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d (s_1) f_k(k; s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \quad (58)
\]

\[
\begin{align*}
&\max_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \geq \\
&\left[ -\min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^d (s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
&- \min_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} + \\
&\max_{i} \left\{ E_{s_0} \left( \frac{f(k; s_1)}{B} \right) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \right\} \right] f_k(\bar{z}^{nd}_1)
\end{align*}
\]

where the term on the lhs is nonnegative because of (56) and the one on the rhs is also nonnegative by construction. Analogously, substituting the expressions for the partial derivatives
into the FOC for $dB = f_k(s_{1,d})dk < 0$ yields:

\[
\left[ -\max_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) \left| s_1 \in S_{1,d} \right. \right\} \Pr\{s_1 \in S_{1,d} \} \\
- \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \left[ \frac{f(k; s_1)}{B} \right] \left| s_1 \in S^d \right. \Pr\{s_1 \in S^d \} + p \right\} f_k(s_{1,d}) + \\
-1 + \min_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) f_k(k, s_1) \right| s_1 \in S_{1,d} \} \Pr\{s_1 \in S_{1,d} \} + \\
\min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \frac{f_k(k; s_1)}{B} \right| s_1 \in S^d \} \Pr\{s_1 \in S^d \} \geq 0
\]

or

\[
\left[ -\max_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) \left| s_1 \in S_{1,d} \right. \right\} \Pr\{s_1 \in S_{1,d} \} \\
- \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \left[ \frac{f(k; s_1)}{B} \right] \left| s_1 \in S^d \right. \Pr\{s_1 \in S^d \} + p \right\} f_k(s_{1,d}) \\
\geq 1 - \min_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) f_k(k, s_1) \right| s_1 \in S_{1,d} \} \Pr\{s_1 \in S_{1,d} \} - \\
\min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \frac{f_k(k; s_1)}{B} \right| s_1 \in S^d \} \Pr\{s_1 \in S^d \}
\]

where the term on the lhs is nonnegative because of (55) and the one on the rhs is also nonnegative as it immediately follows from (56). Putting (58) and (59) together,

\[
1 - \max_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) f_k(k, s_1) \right| s_1 \in S_{1,d} \} \Pr\{s_1 \in S_{1,d} \} \\
- \max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \frac{f_k(k; s_1)}{B} \right| s_1 \in S^d \} \Pr\{s_1 \in S^d \} \geq \\
\left[ -\max_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) \left| s_1 \in S_{1,d} \right. \right\} \Pr\{s_1 \in S_{1,d} \} \\
- \min_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) \frac{f(k; s_1)}{B} \right| s_1 \in S^d \} \Pr\{s_1 \in S^d \} + p \right\} f_k(s_{1,d}) \geq \\
\left[ -\max_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) \left| s_1 \in S_{1,d} \right. \right\} \Pr\{s_1 \in S_{1,d} \} - \\
\max_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \left[ \frac{f(k; s_1)}{B} \right] \left| s_1 \in S^d \right. \Pr\{s_1 \in S^d \} + p \right\} f_k(s_{1,d}) \geq \\
1 - \min_{i \in I^c} E_{s_0} \left\{ \text{MRS}^i(s_1) f_k(k, s_1) \right| s_1 \in S_{1,d} \} \Pr\{s_1 \in S_{1,d} \} - \\
\min_{i \in I^d} E_{s_0} \left\{ \text{MRS}^i(s_1) \frac{f_k(k; s_1)}{B} \right| s_1 \in S^d \} \Pr\{s_1 \in S^d \}
\]
Since

\[-\min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \]

\[-\min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f(k; s_1)}{B} \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \geq 0 \]

\[-\max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \]

\[-\max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \]

and

\[-\min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) f_k(k, s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \]

\[-\min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \geq 0 \]

\[-\max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) f_k(k, s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} \]

\[-\max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \]

it must be that (57) holds, where recall that

\[ p = \max_i \left\{ E_{s_0} \left( \operatorname{MRS}_i(s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} + E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \right\} \]

This implies

\[ \min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} = \]

\[ \max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^d\} = \]

\[ \min_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} = \]

\[ \max_{i \in I^d} E_{s_0} \left( \operatorname{MRS}_i(s_1) \frac{f(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} = \]

59
and

\begin{align*}
\min_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f_k(k, s_1) \right] \right\} s_1 \in S^{n_d} \} \Pr\{s_1 \in S^{n_d} \} = \\
\max_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f_k(k, s_1) \right] \right\} s_1 \in S^{n_d} \} \Pr\{s_1 \in S^{n_d} \}
\end{align*}

\begin{align*}
\min_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i} (s_1) f_k(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \} \\
= \max_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i} (s_1) f_k(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \}
\end{align*}

Note that conditions (55), (56) and (57) can be alternatively stated as:

\begin{align*}
\min_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left| s_1 \in S^{n_d} \right\} \Pr\{s_1 \in S^{n_d} \} + \\
\min_{i \in I^d} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f(k; s_1) \right] \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \} \geq p \geq \\
\max_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left| s_1 \in S^{n_d} \right\} \Pr\{s_1 \in S^{n_d} \} + \\
\max_{i \in I^d} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f(k; s_1) \right] \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \}.
\end{align*}

\begin{align*}
\min_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f_k(k, s_1) \right] \right\} s_1 \in S^{n_d} \} \Pr\{s_1 \in S^{n_d} \} + \\
\min_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i_d} (s_1) f_k(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \} \geq 1 \geq \\
\max_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left[ f_k(k, s_1) \right] \right\} s_1 \in S^{n_d} \} \Pr\{s_1 \in S^{n_d} \} + \\
\max_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i_d} (s_1) f_k(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \}
\end{align*}

and

\begin{align*}
\min_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left| s_1 \in S^{n_d} \right\} \Pr\{s_1 \in S^{n_d} \} = \\
\max_{i \in I^e} E_{s_0} \left\{ \mathcal{MR}^{i_d} (s_1) \left| s_1 \in S^{n_d} \right\} \Pr\{s_1 \in S^{n_d} \}
\end{align*}

\begin{align*}
\min_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i_d} (s_1) f(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \} = \\
\max_{i \in I^d} E_{s_0} \left\{ \frac{\mathcal{MR}^{i_d} (s_1) f(k; s_1)}{B} \right\} s_1 \in S^d \} \Pr\{s_1 \in S^d \}.
\end{align*}
\[
\begin{align*}
\min_{i \in I^e} E_{s_0} \left\{ MRS^d(s_1) f_k(k, s_1) \bigg| s_1 \in S^{nd} \right\} & \Pr\{s_1 \in S^{nd}\} = \\
\max_{i \in I^e} E_{s_0} \left\{ MRS^d(s_1) f_k(k, s_1) \bigg| s_1 \in S^{nd} \right\} & \Pr\{s_1 \in S^{nd}\} \\
+ \min_{i \in I^d} E_{s_0} \left( MRS^d(s_1) \frac{f_k(k, s_1)}{B} \bigg| s_1 \in S^d \right) & \Pr\{s_1 \in S^d\} \\
\max_{i \in I^d} E_{s_0} \left( MRS^d(s_1) \frac{f_k(k, s_1)}{B} \bigg| s_1 \in S^d \right) & \Pr\{s_1 \in S^d\}
\end{align*}
\]

This completes the proof of the claim. \[\square\]