G25.2651: Statistical Mechanics

Problem set #5

Due April 20, 2009

1. Consider two distinguishable particles in one dimension with respective coordinates x and y and conjugate momenta p_x and p_y with a Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2M} + U(x) + \frac{1}{2}M\omega^2 y^2 - \lambda xy$$

   a. Show that the density matrix $\rho(x, y, x', y'; \beta)$ can be written in the form

$$\rho(x, y, x', y'; \beta) = \int_{x(0)=x}^{x(\beta\hbar)=x'} \mathcal{D}x(\tau) \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left(\frac{1}{2}m\dot{x}^2(\tau) + U(x(\tau))\right)\right] T[x; y, y']$$

   where $T[x; y, y']$ is known as the *influence functional*. What is the functional integral expression for $T[x; y, y']$, and of what function is $T[x; y, y']$ a functional?

   b. Give a closed form expression for $T(x(\tau), y, y')$ by evaluating the functional integral.
   <u>Hint</u>: Consider Using the method of expansion about the classical path.

2. Consider, again, the finite-P expression $Q_P(\beta)$ for the partition function for a particle moving in one dimension.

$$Q_P(\beta) = \left(\frac{mP}{2\pi\beta\hbar^2}\right)^{P/2} \int dx_1 \cdots dx_P \; \exp\left\{-\sum_{i=1}^{P}\left[\frac{mP}{2\beta\hbar^2}(x_i - x_{i+1})^2 + \frac{\beta}{P}U(x_i)\right]\right\}$$

   Using the thermodynamic relation for the constant-volume heat capacity

$$C_V = k\beta^2 \frac{\partial^2}{\partial\beta^2} \ln Q(\beta)$$

   derive primitive and virial estimators for C_V. Be sure to show **all** of your work.

3. a. Two identical, non-interacting fermions of mass m are in a harmonic oscillator potential $U(x) = m\omega x^2/2$, where ω is the oscillator frequency. Calculate the canonical partition function of the system at temperature T.

   b. Repeat for two identical, non-interacting bosons.