Problem set #1
Due: Feb. 4, 2016

1. **Nonuniqueness of the Lagrangian**: The Lagrangian of a system of \(N\) particles in Cartesian coordinates is

\[
L(r, \dot{r}) = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2 - U(r_1, ..., r_N)
\]

Let \(F(r_1, ..., r_N, t)\) be any differentiable function of the coordinates and of time. Show that the Lagrangian

\[
L'(r, \dot{r}, t) = L(r, \dot{r}) + \frac{dF(r_1, ..., r_N, t)}{dt}
\]

gives the same equations of motion as \(L(r, \dot{r})\).

2. It has been suggested that when a system is subject to a shearing force, its Hamiltonian should be modified to read

\[
H(r, p) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + U(r_1, ..., r_N) + \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} r_{i,\alpha} B_{\alpha\beta} p_{i,\beta}
\]

where \(\alpha\) and \(\beta\) index the three spatial components of the vectors \(r_i\) and \(p_i\) such that \(r_{i,1} = x_i, r_{i,2} = y_i, r_{i,3} = z_i\) with analogous identifications for \(p_{i,\beta}\). \(B_{\alpha\beta}\) is a constant matrix.

a. Derive Hamilton’s equations of motion for this Hamiltonian.

b. Suppose the elements of the matrix \(B_{\alpha\beta}\) are \(B_{32} = \gamma\) and \(B_{\alpha\beta} = 0\) otherwise. Here \(\gamma\) is a constant. Examine the \(\dot{r}_i = \partial H / \partial p_i\) equation carefully in the limit that \(p_i \to 0\) and show that the \(r_{i,\alpha} B_{\alpha\beta} p_{i,\beta}\) term does, indeed, produce a shearing effect. **Hint**: One way to do this is to plot the component of \(\dot{r}_i\) that is affected by this term as a function of \(r_i\) in the limit that \(p_i \to 0\).

3. A particle of mass \(m\) with coordinate \(x\) and momentum \(p\) moves in a double-well potential of the form

\[
U(x) = \frac{U_0}{a^4} (x^2 - a^2)^2.
\]

Sketch the contours of the constant-energy surface \(H(x, p) = E\) in phase space for the following cases:

a. \(E < U_0\).

b. \(E = U_0 + \epsilon\), where \(\epsilon \ll U_0\).

c. \(E > U_0\).

4. Consider a system with coordinate \(q\), momentum \(p\), and Hamiltonian

\[
H = \frac{p^n}{n} + \frac{q^n}{n},
\]
where \( n \) is an integer larger than 2. Show that if the energy \( E \) of the system is chosen such that \( nE = m^n \), where \( m \) is a positive integer, then no phase space trajectory can ever pass through a point for which \( p \) and \( q \) are both positive integers. Consider a system with coordinate \( q \), momentum \( p \), and Hamiltonian

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**Hint:** You might find Fermat’s last theorem helpful here.

5. Consider an ensemble of one-particle systems, each evolving in one spatial dimension according to an equation of motion of the form

\[
\dot{x} = -\alpha x
\]

where \( \alpha > 0 \), and where \( x(t) \) is the particle position at time \( t \). The Liouville equation for the ensemble distribution \( f(x, t) \) is

\[
\frac{\partial f}{\partial t} - \alpha x \frac{\partial f}{\partial x} = \alpha f
\]

a. Suppose that at \( t = 0 \), the ensemble distribution is given by

\[
f(x, 0) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + x^2}
\]

where \( \sigma \) is a constant. Find the ensemble distribution \( f(x, t) \) at all time, and discuss the behavior of this distribution as \( t \to \infty \). Finally, show that your distribution function \( f(x, t) \) is normalized for all \( t \).

**Hint:** Show that the substitution \( f(x, t) = e^{\alpha t} \tilde{f}(x, t) \) yields an equation for a conserved distribution \( \tilde{f}(x, t) \). Next, try multiplying the \( x \) in the initial distribution by a function \( g(t) \), where \( g(0) = 1 \), and use the Liouville equation to derive an equation that \( g(t) \) must satisfy.

b. Repeat for the case that the initial distribution is a Gaussian,

\[
f(x, 0) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-x^2/2\sigma^2}
\]

where \( \sigma \) is a constant.