

G25.2651: Statistical Mechanics

Notes for Lecture 28

I. FIXED POINTS OF THE RG EQUATIONS IN GREATER THAN ONE DIMENSION

In the last lecture, we noted that interactions between block of spins in a spin lattice are mediated by boundary spins. In one dimension, where there is only a single pair between blocks, the block spin transformation yields a coupling constant that is approximately equal to the old coupling constant at very low temperature, i.e., $K' \sim K$. Let us now explore the implications of this fact in higher dimensions.

Consider a two-dimensional spin lattice as we did in the previous lecture. Now interactions between blocks can be mediated by more than a single pair of spin interactions. For the case of 3×3 blocks, there will be 3 boundary spin pairs mediating the interaction between two blocks:

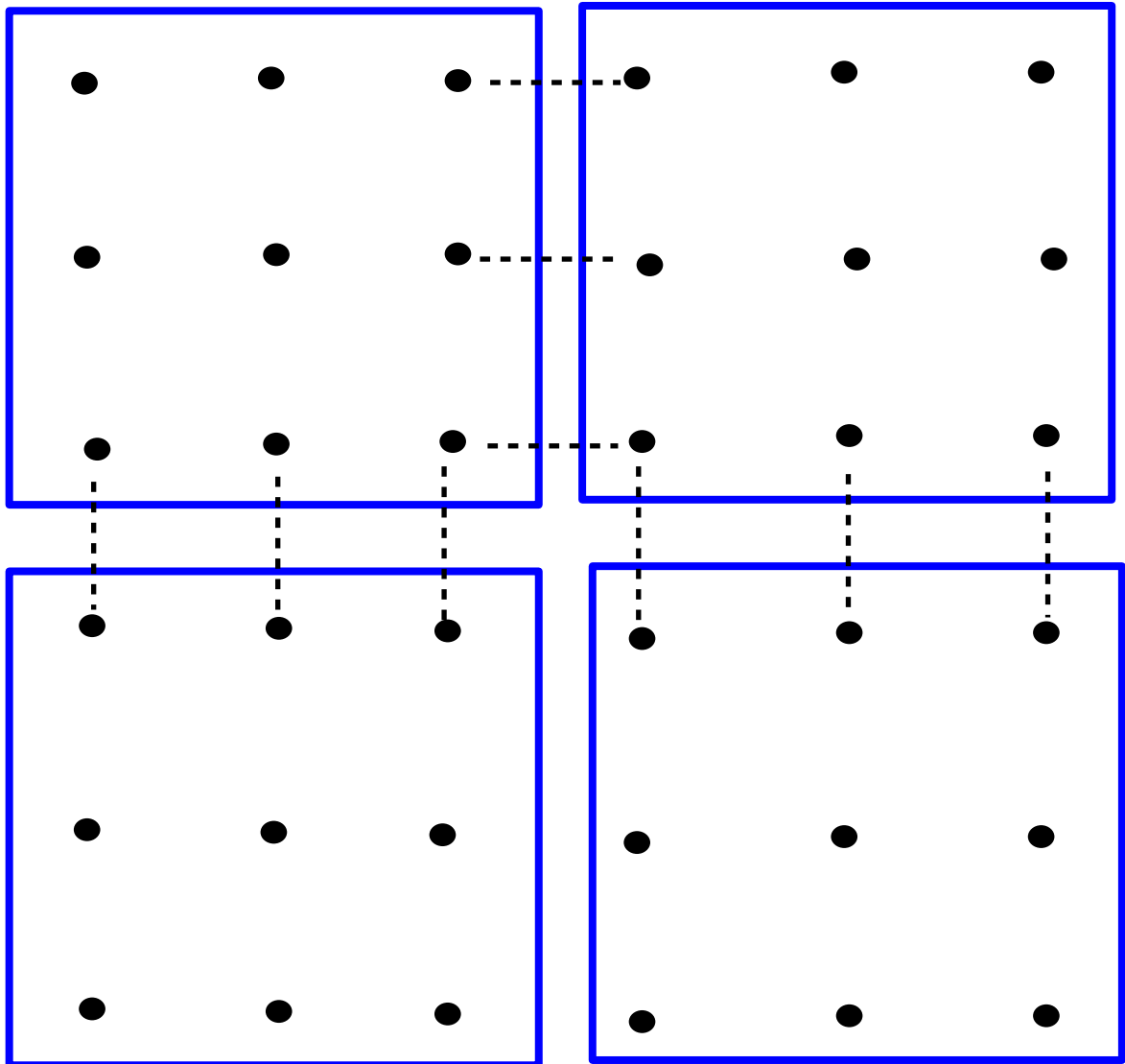


FIG. 1.

Since 3 boundary spin pair interactions mediate the block-block interaction, the result of a block spin transformation should yield, at low T , a coupling constant K' roughly three times as large as the original coupling constant, K :

$$K' \sim 3K$$

In a three-dimensional lattice, using $3 \times 3 \times 3$ blocks, there would be $3^2 = 9$ spin pairs. Generally, in d dimensions using blocks containing b^d spins, the RG equation at low T should behave as

$$K' \sim b^{d-1}K \quad T \rightarrow 0, K \rightarrow \infty$$

The number b is called the *length scaling factor*. The above RG equation implies that for $d > 1$, $K' > K$ for low T . Thus, iteration of the RG equation at low temperature should now flow toward $K = \infty$ and the fixed point at $T = 0$ is now a stable fixed point. However, we know that at high temperature, the system must be in a paramagnetic state, so the fixed point at $T = \infty$ must remain a stable fixed point. These two facts suggest that, for $d > 1$, between $T = 0$ and $T = \infty$, there must be another fixed point, which will be an unstable fixed point. To the extent that an RG flow in more than one dimension can be considered a one-dimensional flow, the flow diagram would look like:

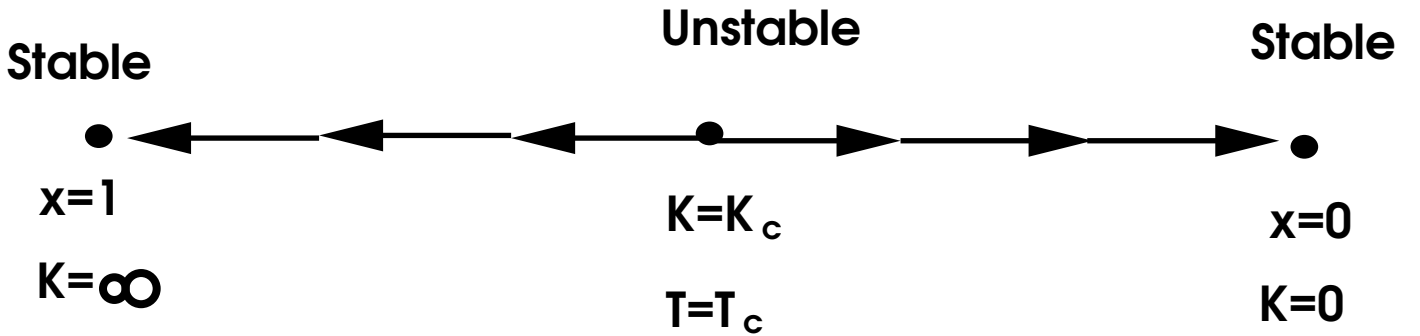


FIG. 2.

Any perturbation to the left of the unstable fixed point iterates to $T = 0$, $K = \infty$ and any perturbation to the right iterates to $T = \infty$ and $K = 0$. The unstable fixed point corresponds to a finite, nonzero value of $K = K_c$ and a temperature T_c , and corresponds to a critical point.

To see that this is so, consider the correlation length evolution of the correlation length under the RG flow. Recall that for a length scaling factor b , the correlation length transform as

$$\begin{aligned} \xi(K') &= \frac{1}{b}\xi(K) \\ \xi(K) &= b\xi(K') \end{aligned}$$

Suppose we start at a point K near K_c and require $n(K)$ iterations of the RG equations to reach a value K_0 between $K = 0$ and $K = K_c$ under the RG flow:

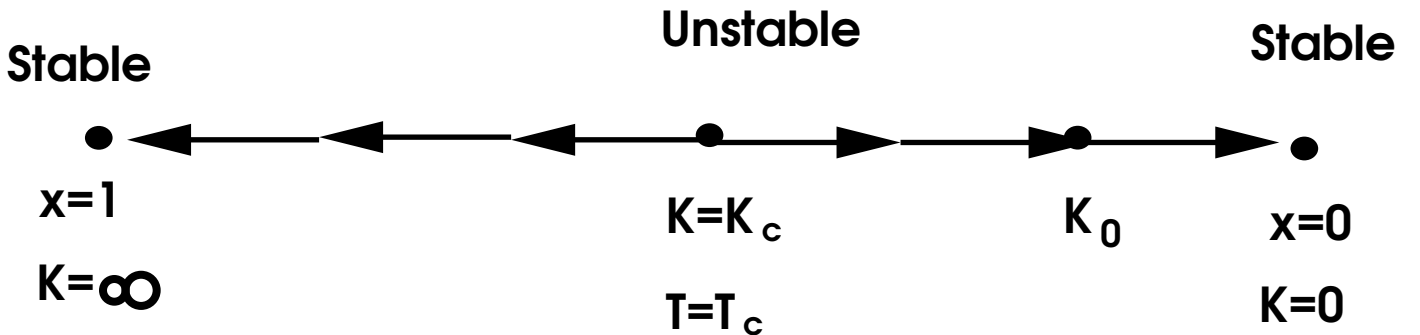


FIG. 3.

If ξ_0 is the correlation length at $K = K_0$, which can expect to be a finite number of order 1, then, by the above transformation rule for the correlation length, we have

$$\xi(K) = \xi_0 b^{n(K)}$$

Now, recall that, as the starting point, K is chosen closer and closer to K_c , the number of iterations needed to reach K_0 gets larger and larger. (Recall that near an unstable fixed point, the initial change in the coupling constant is small as the iteration proceeds). Of course, if $K = K_c$ initially, than an infinite number of iterations is needed. This tells us that as K approaches K_c , the correlation length becomes infinite, which is what is expected for an ordered phase. Thus, the new unstable fixed point must correspond to a critical point.

In fact, we can calculate the exponent ν knowing the behavior of the RG equation near the unstable fixed point. Since this is a fixed point, K_c satisfies, quite generally,

$$K_c = R(K_c)$$

Near the fixed point, we can expand the RG equation, giving

$$K' \approx R(K_c) + (K - K_c)R'(K_c) + \dots$$

Define an exponent y by

$$y = \frac{\ln R'(K_c)}{\ln b}$$

so that

$$K' \approx K_c + b^y(K - K_c)$$

Near the critical point, ξ diverges according to

$$\xi \sim |T - T_c|^{-\nu} \sim \left| \frac{1}{K} - \frac{1}{K_c} \right|^{-\nu} \sim \left| \frac{K - K_c}{K} \right|^{-\nu} \sim \left| \frac{K - K_c}{K_c} \right|^{-\nu}$$

Thus,

$$\xi \sim |K - K_c|^{-\nu}$$

but

$$\xi(K) = b\xi(K')$$

which implies

$$\begin{aligned} |K - K_c|^{-\nu} &\sim b|K' - K_c|^{-\nu} \\ &= b|b^y(K - K_c)|^{-\nu} \end{aligned}$$

which is only possible if

$$\nu = \frac{1}{y}$$

This result illustrates a more general one, namely, that critical exponents are related to derivatives of the RG transformation.

II. GENERAL LINEARIZED RG THEORY

The above discussion illustrates the power of the linearized RG equations. We now generalize this approach to a general Hamiltonian \mathcal{H}_0 with parameters $K_1, K_2, \dots, \equiv \mathbf{K}$. The RG equation

$$\mathbf{K}' = \mathbf{R}(\mathbf{K})$$

can be linearized about an unstable fixed point at bK^* according to

$$K'_a - K_a^* \approx \sum_b T_{ab}(K_b - K_b^*)$$

where

$$T_{ab} = \left. \frac{\partial R_a}{\partial K_b} \right|_{\mathbf{K}=\mathbf{K}^*}$$

The matrix T need not be a symmetric matrix. Given this, we define a left eigenvalue equation for T according to

$$\sum_a \phi_a^i T_{ab} = \lambda^i \phi_b^i$$

where the eigenvalues $\{\lambda\}$ can be assumed to be real (although it cannot be proved). Finally, define a *scaling variable*, u_i by

$$u_i = \sum_a \phi_a^i (K_a - K_a^*)$$

They are called scaling variables because they transform multiplicatively near a fixed point under the linearized RG flow:

$$\begin{aligned} u'_i &= \sum_a \phi_a^i (K'_a - K_a^*) \\ &= \sum_a \sum_b \phi_a^i T_{ab} (K_b - K_b^*) \\ &= \sum_b \lambda^i \phi_b^i (K_b - K_b^*) \\ &= \lambda^i u_i \end{aligned}$$

Since u_i scales with λ^i , it will increase if $\lambda^i > 1$ and will decrease if $\lambda^i < 1$. Redefining the eigenvalues λ^i according to

$$\lambda^i = b^{y_i}$$

we see that

$$u'_i = b^{y_i} u_i$$

By convention, the quantities $\{y_i\}$ are called the RG eigenvalues. These will soon be shown to determine the scaling relations among the critical exponents.

For the RG eigenvalues, three cases can be identified:

1. $y_i > 0$. The scaling variable u_i is called a *relevant* variable. Repeated RG transformations will drive it away from its fixed point value, $u_i = 0$.
2. $y_i < 0$. The scaling variable u_i is called an *irrelevant* variable. Repeated RG transformations will drive it toward 0.
3. $y_i = 0$. The scaling variable u_i is called a *marginal* variable. We cannot tell from the linearized RG equations if u_i will iterate toward or away from the fixed point.

Typically, scaling variables are either relevant or irrelevant. Marginality is rare and will not be considered here. The number of relevant scaling variables corresponds to the number of experimentally ‘tunable’ parameters or ‘knobs’ (such as T and h in the magnetic system, or P and T in a fluid system; in the case of the former, the relevant variables are called the *thermal* and *magnetic* scaling variables, respectively).

III. UNDERSTANDING UNIVERSALITY FROM THE LINEARIZED RG THEORY

In the linearized RG theory, at a fixed point, all scaling variables are 0, whether relevant, irrelevant or marginal. Consider the case where there are no marginal scaling variables. Recall, moreover, that irrelevant scaling variables will iterate to 0 under repeated RG transformations, starting from a point near the unstable fixed point, while the relevant variables will be driven away from 0. These facts provide us with a formal procedure for locating the fixed point:

- i. Start with the space spanned by the full set of eigenvectors of T .
- ii. Project out the relevant subspace by setting all the relevant scaling variables to 0 by hand.
- iii. The remaining subspace spanned by the irrelevant eigenvectors of T defines a hypersurface in the full coupling constant space. This is called the critical hypersurface.
- iv. Any point on the critical hypersurface belongs to the irrelevant subspace and will iterate to 0 under successive RG transformations. This will define a trajectory on the hypersurface that leads to the fixed point as illustrated below:

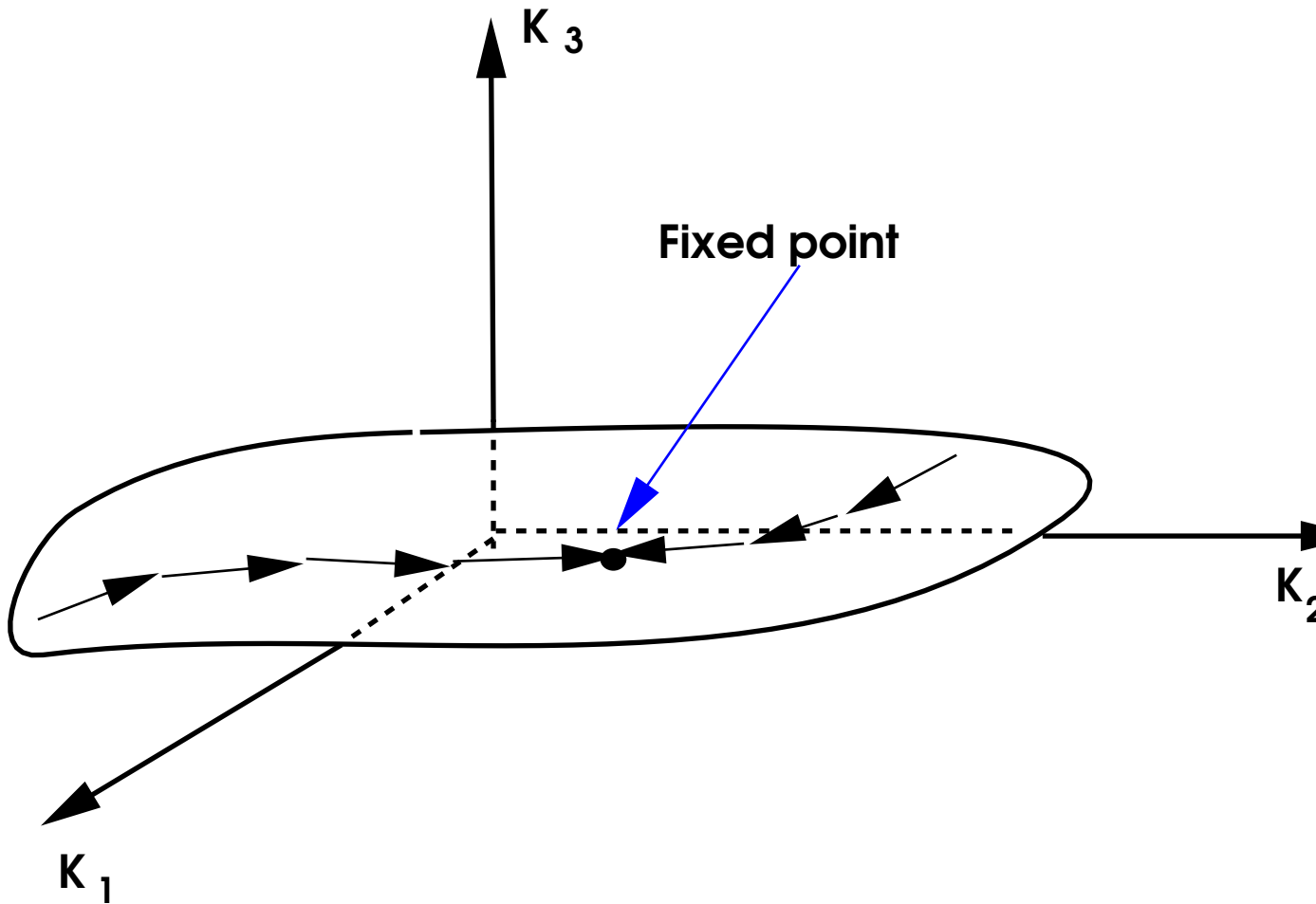


FIG. 4.

This fixed point is called the *critical fixed point*. Note that it is stable with respect to irrelevant scaling variables and unstable with respect to relevant scaling variables.

What is the importance of the critical fixed point? Consider a simple model in which there is one relevant and one irrelevant scaling variable, u_1 and u_2 , with corresponding couplings K_1 and K_2 . In an Ising-type model, K_1 might represent the reduced nearest neighbor coupling and K_2 might represent a next nearest neighbor coupling. Relevant variables also include experimentally tunable parameters such as temperature and magnetic field. The reason u_1 is relevant and u_2 is irrelevant is that there must be at least nearest neighbor coupling for the existence of a critical point and ordered phase at $h = 0$ but this can happen whether or not there is a next nearest neighbor coupling. Thus, the condition

$$u_1(K_1, K_2) = 0$$

defines the critical surface, in this case, a one-dimensional curve in the K_1 - K_2 plane as illustrated below:

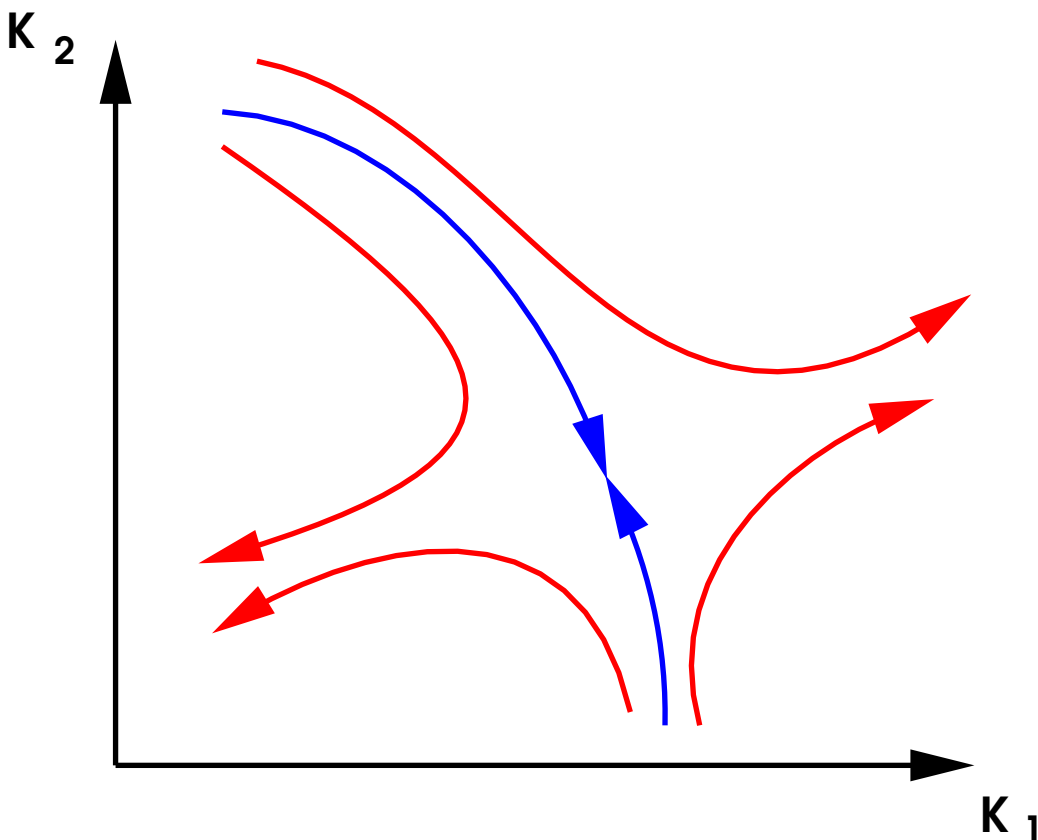


FIG. 5.

Here, the blue curve represents the critical “surface” (curve), and the point where the arrows meet is the critical fixed point. The full coupling constant space represents the space of all physical systems containing nearest neighbor and next nearest neighbor couplings. If we wish to consider the subset of systems with no next nearest neighbor coupling, i.e., $K_2 = 0$, the point at which the line $K_2 = 0$ intersects the critical surface (curve) defines the critical value, K_{1c} and corresponding critical temperature and will be an unstable fixed point of an RG transformation with $K_2 = 0$. Similarly, if we consider a model for which $K_2 \neq 0$, but having a fixed finite value, then the point at which this line intersects the critical surface (curve) will give the critical value of K_1 for that model. For any of these models, K_{1c} lies on the critical surface and will, under the full RG transformation iterate toward the critical fixed point. This, then, defines a universality class: All models characterized by the same critical fixed point belong to the same universality class and will share the same critical properties. This need not include only magnetic systems. Indeed, a fluid system, near its critical point can be characterized by a so called *lattice gas model*. This model is similar to the Ising model, except that the spin variables are replaced by site occupancy variables n_i , which can take on values 0 or 1, depending on whether a given lattice site is occupied by a particle or not. The grand canonical partition function is

$$Z = \sum_{n_1=0,1} \cdots \sum_{n_N=0,1} \zeta^{\sum_i n_i} e^{2\beta \sum_{\langle i,j \rangle} J_{ij} n_i n_j}$$

and hence belongs to the same universality class as the Ising model. The critical surface, then, contains all physical models that share the same universality properties and have the same critical fixed point.

In order to see how the relevant scaling variables lead to the scaling relations among the critical exponents, we next need to introduce the scaling hypothesis.

IV. THE SCALING HYPOTHESIS

Recall that the RG transformation preserves the partition function:

$$\mathrm{Tr}_{\sigma'} e^{-\mathcal{H}'_0(\{\sigma'\}, K')} = \mathrm{Tr}_{\sigma} e^{-\mathcal{H}_0(\{\sigma\}, K)}$$

For the one-dimensional Ising model, we found that the block spin transformation lead to a transformed Hamiltonian of the form

$$\mathcal{H}'_0(\{\sigma'\}, K') = N' g(K) - K' \sum_{i=1}^{N'} \sigma'_i \sigma'_{i+1}$$

Thus, $\mathcal{H}_0(\{\sigma\}, K)$ contains a term that is of the same functional form as $\mathcal{H}'_0(\{\sigma'\}, K')$ plus an analytic function of K . Defining the reduced free energy per spin from the spin trace as $f(K)$, the equality of the partition functions allows us to write generally:

$$e^{-Nf(\{K\})} = \mathrm{Tr}_{\sigma} e^{-\mathcal{H}_0(\{\sigma\}, \{K\})} = e^{-N'g(\{K\})} \mathrm{Tr}_{\sigma'} e^{-\mathcal{H}'_0(\{\sigma'\}, \{K'\})} = e^{-N'g(\{K\})} e^{-N'f(\{K'\})}$$

which implies that

$$f(\{K\}) = g(\{K\}) + \frac{N'}{N} f(\{K'\})$$

If b is the size of the spin block, then

$$N' = b^{-d} N$$

from which

$$f(\{K\}) = g(\{K\}) + b^{-d} f(\{K'\})$$

Now, $g(\{K\})$ is an analytic function and therefore plays no role in determining critical exponents, since it does not lead to divergences. Only the so called *singular* part of the free energy is important for this. Thus, the singular part of the free energy $f_s(\{K\})$ can be seen to satisfy a *scaling relation*:

$$f_s(\{K\}) = b^{-d} f_s(\{K'\})$$

This is the basic scaling relation for the free energy. From this simple equation, the scaling relations for the critical exponents can be derived.

To see how this works, consider the one-dimensional Ising model again with $h \neq 0$. The free energy depends on the scaling variables through the dependence on the couplings K . For $h = 0$, we saw that there was a single relevant scaling variable corresponding to the nearest neighbor coupling $K = J/kT$. This variable is temperature dependent and is called a *thermal scaling variable* u_t . For $h \neq 0$ there must also be a magnetic scaling variable, u_h . These will transform under the linearized RG as

$$\begin{aligned} u'_t &= b^{y_t} u_t \\ u'_h &= b^{y_h} u_h \end{aligned}$$

where y_t and y_h are the relevant RG eigenvalues. Therefore, the scaling relation for the free energy becomes

$$f_s(u_t, u_h) = b^{-d} f_s(u'_t, u'_h) = b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h)$$

Now, after n iterations of the RG equations, the free energy becomes

$$f_s(u_t, u_h) = b^{-nd} f_s(b^{ny_t} u_t, b^{ny_h} u_h)$$

Recall that relevant scaling variables are driven away from the critical fixed point. Thus, let us choose an n small enough that the linear approximation is still valid. In order to determine n , we only need to consider one of the scaling variables, so let it be u_t . Thus, let u_{t0} be an arbitrary value of u_t obtained after n iterations of the RG equation, such that u_{t0} is still close enough to the fixed point that the linearized RG theory is valid. Then,

$$u_{t0} = b^{ny_t} u_t$$

or

$$n = \frac{1}{y_t} \log_b \left| \frac{u_{t0}}{u_t} \right| = \log_b \left| \frac{u_{t0}}{u_t} \right|^{1/y_t}$$

and

$$f_s(u_t, u_h) = \left| \frac{u_t}{u_{t0}} \right|^{d/y_t} f_s(\pm u_{t0}, u_h |u_t/u_{t0}|^{-y_h/y_t})$$

Now, let

$$t = \frac{T - T_c}{T}$$

We know that f_s must depend on the physical variables t and h . In the linearized theory, the scaling variables u_t and u_h will be related linearly to the physical variables:

$$\begin{aligned} \frac{u_t}{u_{t0}} &= \frac{t}{t_0} \\ u_h &= \frac{h}{h_0} \end{aligned}$$

Here, t_0 and h_0 are nonuniversal proportionality constants, and we see that $u_t \rightarrow 0$ when $t \rightarrow 0$ and the same for u_h . Then, we have

$$f_s(t, h) = |t/t_0|^{d/y_t} f_s\left(\pm u_{t0}, \frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)$$

The left side of this equation does not depend on the nonuniversal constant u_{t0} , hence the right side cannot. This means that the function on the right side depends on a single argument. Thus, we rewrite the free energy equation as

$$f_s(t, h) = |t/t_0|^{d/y_t} \Phi\left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)$$

The function Φ is called a *scaling function*. Note that the dependence on the system-particular variables t_0 and h_0 is trivial, as this only comes in as a scale factor in t and h . Such a scaling relation is a universal scaling relation, in that it will be the same for all systems in the same universality class. From the above scaling relation come all thermodynamic quantities. Note that the scaling relation depends on only two exponents y_t and y_h . This suggests that there can only be two independent critical exponents among the six, $\alpha, \beta, \gamma, \delta, \nu$, and η . There must, therefore, be four relations relating some of the critical exponents to others. These are the *scaling relations*. To derive these, we use the scaling relation for the free energy to derive the thermodynamic functions:

1. Heat Capacity:

$$C_h \sim \left. \frac{\partial^2 f}{\partial t^2} \right|_{h=0} \sim |t|^{d/y_t - 2}$$

but

$$C_h \sim |t|^{-\alpha}$$

Thus,

$$\alpha = 2 - \frac{d}{y_t}$$

2. Magnetization:

$$m = \left. \frac{\partial f}{\partial h} \right|_{h=0} \sim \frac{|t/t_0|^{d/y_t}}{|t/t_0|^{y_h/y_t}} \sim |t|^{(d-y_h)/y_t}$$

but

$$m \sim |t|^\beta$$

Thus,

$$\beta = \frac{d - y_h}{y_t}$$

3. Magnetic susceptibility:

$$\chi = \frac{\partial m}{\partial h} \sim \frac{\partial^2 f}{\partial h^2} \sim |t|^{(d-2y_h)/y_t}$$

but

$$\chi \sim |t|^{-\gamma}$$

Thus,

$$\gamma = \frac{2y_h - d}{y_t}$$

4. Magnetization vs. magnetic field:

$$m = \frac{\partial f}{\partial h} = |t/t_0|^{(d-y_h)/y_t} \Phi' \left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}} \right)$$

but m should remain finite as $t \rightarrow 0$, which means that $\Phi'(x)$ must behave as x^{d/y_h-1} as $x \rightarrow \infty$, since then

$$\begin{aligned} m &\sim |t/t_0|^{(d-y_h)/y_t} \frac{(h/h_0)^{(d-y_h)/y_t}}{|t/t_0|^{y_h(d-y_h)/(y_t y_h)}} \\ &\sim (h/h_0)^{(d-y_h)/y_h} \end{aligned}$$

But

$$m \sim h^{1/\delta}$$

Thus,

$$\delta = \frac{y_h}{d - y_h}$$

From these, it is straightforward to show that the four exponents α , β , γ , and δ are related by

$$\begin{aligned}\alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2\end{aligned}$$

These are examples of *scaling relations* among the critical exponents.

The other two scaling relations involve ν and η and are derived from the scaling relation satisfied by the correlation function:

$$\begin{aligned}G(r) &= b^{-2(d-y_h)} G(r/b, b^{y_t} t) \\ G(r) &= |t/t_0|^{2(d-y_h)/y_t} \Psi\left(\frac{r}{|t/t_0|^{-1/y_t}}\right)\end{aligned}$$

This leads to the relations

$$\begin{aligned}\nu &= \frac{1}{y_t} \\ \eta &= d + 2 - 2y_h\end{aligned}$$

and the scaling relations:

$$\begin{aligned}\alpha &= 2 - d\nu \\ \gamma &= \nu(2 - \eta)\end{aligned}$$