G25.2666: Quantum Mechanics II

Notes for Lecture 7

I. A SIMPLE EXAMPLE OF ANGULAR MOMENTUM ADDITION

Given two spin-1/2 angular momenta, $\mathbf{S}_1$ and $\mathbf{S}_2$, we define

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

The problem is to find the eigenstates of the total total spin operators S^2 and S_z and identify the allowed total spin states.

In order to solve this problem, we recognize that the eigenvectors we seek will be expressible in terms of tensor products of the spin eigenstates of the individual spins. There will be four such vectors:

$$\begin{aligned}
\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle &= \left|\frac{1}{2}\quad\frac{1}{2}\right\rangle\bigotimes\left|\frac{1}{2}\quad\frac{1}{2}\right\rangle \\
\left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle &= \left|\frac{1}{2}\quad-\frac{1}{2}\right\rangle\bigotimes\left|\frac{1}{2}\quad\frac{1}{2}\right\rangle \\
\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle &= \left|\frac{1}{2}\quad\frac{1}{2}\right\rangle\bigotimes\left|\frac{1}{2}\quad-\frac{1}{2}\right\rangle \\
\left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle &= \left|\frac{1}{2}\quad-\frac{1}{2}\right\rangle\bigotimes\left|\frac{1}{2}\quad-\frac{1}{2}\right\rangle
\end{aligned}$$

Although this is a valid set of basis vectors for the full spin Hilbert space, these are *not* eigenvectors of S^2 or S_z. Thus, we seek a unitary transformation among these vectors that generates a new set of four vectors that *are* eigenvectors of S^2 and S_z.

The four new vectors will correspond to adding the spins in either a parallel or anti-parallel fashion:

Parallel	$\uparrow\uparrow$	Total Spin = 1,	$m_s = -1, 0, 1$	=	3 states
Antiparallel	$\uparrow\downarrow$	Total Spin = 0,	$m_s = 0$	=	1 state

for a total of four states, as expected.

To find these states, begin by noting that the state

$$\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle$$

is, in fact, an eigenstate of S^2 and S_z. To see this note that

$$\begin{aligned}
S_z\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle &= (S_{1z}+S_{2z})\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle \\
&= \frac{\hbar}{2}\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle + \frac{\hbar}{2}\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle \\
&= \hbar\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle
\end{aligned}$$

Also, since

$$\begin{aligned}
S^2 &= (\mathbf{S}_1+\mathbf{S}_2)^2 \\
&= S_1^2 + S_2^2 + 2\mathbf{S}_1\cdot\mathbf{S}_2 \\
&= S_1^2 + S_2^2 + 2(S_{1x}S_{2x}+S_{1y}S_{2y}+S_{1z}S_{2z}) \\
&= S_1^2 + S_2^2 + 2S_{1z}S_{2z} + (S_{1+}S_{2-}+S_{1-}S_{2+})
\end{aligned}$$

The action of S^2 on $\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$ is

$$S^2\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle = \left[S_1^2 + S_2^2 + 2S_{1z}S_{2z}\right]\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle + \left(S_{1+}S_{2-} + S_{1-}S_{2+}\right)\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$$

However, the second term on the right vanishes because each involves a raising operator acting on a state $\left|\frac{1}{2}\ \ \frac{1}{2}\right\rangle$ for either the first or second spin, which annihilates the state. Thus,

$$\begin{aligned}
S^2\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle &= \left[S_1^2 + S_2^2 + 2S_{1z}S_{2z}\right]\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle \\
&= \left[\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + 2\frac{\hbar}{2}\frac{\hbar}{2}\right]\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle \\
&= \hbar^2\left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right)\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle \\
&= 2\hbar^2\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle
\end{aligned}$$

Thus, the eigenvalue of S^2 is $2\hbar^2 = 1(1+1)\hbar^2$, corresponding to a value of $s = 1$ with a corresponding S_z value of $\hbar$. These facts make it clear that $\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$ is a total spin-1 state $|s\ \ m_s\rangle = |1\ \ 1\rangle$ of the total spin:

$$|1\ \ 1\rangle = \left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$$

In order to find the the states corresponding to $m_s = 0$ and $m_s = -1$, we can use the total lowering operator:

$$S_- = S_{1-} + S_{2-}$$

Recall the general relation for the action of raising and lowering operators on general spin states:

$$S_\pm|s\ \ m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}|s\ \ m_s \pm 1\rangle$$

The procedure is, then, to act on both sides of

$$|1\ \ 1\rangle = \left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$$

with S_-:

$$S_-|1\ \ 1\rangle = S_-\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle = (S_{1-} + S_{2-})\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle$$

On the left, we obtain $\sqrt{2}\hbar|1\ \ 0\rangle$ so that

$$\begin{aligned}
\sqrt{2}\hbar|1\ \ 0\rangle &= (S_{1-} + S_{2-})\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle \\
&= \hbar\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle + \hbar\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle \\
|1\ \ 0\rangle &= \frac{1}{\sqrt{2}}\left[\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle\right]
\end{aligned}$$

Finally, the state with $m_s = -1$ is obtained by acting again with the lowering operator:

$$\begin{aligned}
S_-|1\ \ 0\rangle &= \frac{1}{\sqrt{2}}S_-\left[\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle\right] = \frac{1}{\sqrt{2}}\left(S_{1-} + S_{2-}\right)\left[\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle + \left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle\right] \\
\sqrt{2}\hbar|1\ \ -1\rangle &= \frac{1}{\sqrt{2}}\left[S_{1-}\left|\frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle + S_{2-}\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right\rangle\right] \\
2\hbar|1\ \ -1\rangle &= \hbar\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle + \hbar\left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle \\
|1\ \ -1\rangle &= \left|\frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right\rangle
\end{aligned}$$

The last state is $|0\ \ 0\rangle$ and is obtained by recognizing that it must be composed of those states that have opposite values of m_{s_1} and m_{s_2}. So we include these states with arbitrary coefficients:

$$|0\ \ 0\rangle = \alpha\left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle + \beta\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle$$

The coefficients α and β are then determined by the conditions that $|0\ \ 0\rangle$ must be normalized and that it must be orthogonal to the other three states. The first condition yields:

$$\langle 0\ \ 0|0\ \ 0\rangle = \alpha^2 + \beta^2 = 1$$

In order to fulfill the second, we recognize that $|0\ \ 0\rangle$ is manifestly orthogonal to $|1\ \ 1\rangle$ and $1\ \ -1\rangle$. However, orthogonality to $|1\ \ 0\rangle$ needs to be enforced:

$$\langle 1\ \ 0|0\ \ 0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(\left\langle\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right| + \left\langle\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right|\right)\quad\left(\alpha\left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle + \beta\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle\right) = 0$$

$$\frac{1}{\sqrt{2}}(\alpha+\beta) = 0$$

$$\Rightarrow \qquad \alpha = -\beta$$

From the normalization condition, it is clear, then, that $\alpha = \pm 1/\sqrt{2}$ so that $\beta = \mp 1/\sqrt{2}$. The choice of sign is arbitrary, so we choose $\alpha = -1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$ so that

$$|0\ \ 0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle - \left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle\right)$$

Thus, the four new basis vectors are:

$$\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle,\qquad \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle + \left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle\right),\qquad \left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle$$
$$\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\right\rangle - \left|\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right\rangle\right)$$

We can write the transformation from the old basis to the new basis as a matrix equation:

$$\begin{pmatrix} |1\ \ 1\rangle \\ |1\ \ 0\rangle \\ |1\ \ -1\rangle \\ |0\ \ 0\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} |\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\rangle \\ |\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\rangle \\ |\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\rangle \\ |\frac{1}{2}\quad -\frac{1}{2};\frac{1}{2}\quad -\frac{1}{2}\rangle \end{pmatrix}$$

The matrix

$$\mathrm{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

can easily be shown to be unitary. Note that the elements of the matrix can be computed from the following overlaps:

$$\left\langle\frac{1}{2}\quad \frac{1}{2};\frac{1}{2}\quad \frac{1}{2}\right|1\ \ 1\rangle = 1$$

$$\left\langle \frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right|1\ \ 0\rangle = \frac{1}{\sqrt{2}}$$
$$\left\langle \frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right|1\ \ 0\rangle = \frac{1}{\sqrt{2}}$$
$$\left\langle \frac{1}{2}\ \ \frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right|0\ \ 0\rangle = \frac{1}{\sqrt{2}}$$
$$\left\langle \frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ \frac{1}{2}\right|0\ \ 0\rangle = -\frac{1}{\sqrt{2}}$$
$$\left\langle \frac{1}{2}\ \ -\frac{1}{2};\frac{1}{2}\ \ -\frac{1}{2}\right|1\ \ -1\rangle = 1$$

These are examples of what are known as *Clebsch-Gordan* coefficients. In the next section, we will study the general forms of these coefficients.

Finally, note that the new basis vectors are, in fact, eigenvectors of S^2 and S_z. Thus, they satisfy

$$\begin{aligned} S^2|1\ \ 1\rangle &= 2\hbar^2|1\ \ 1\rangle & S_z|1\ \ 1\rangle &= \hbar|1\ \ 1\rangle \\ S^2|1\ \ 0\rangle &= 2\hbar^2|1\ \ 0\rangle & S_z|1\ \ 0\rangle &= 0 \\ S^2|1\ \ -1\rangle &= 2\hbar^2|1\ \ -1\rangle & S_z|1\ \ -1\rangle &= -\hbar|1\ \ -1\rangle \\ S^2|0\ \ 0\rangle &= 0 & S_z|0\ \ 0\rangle &= 0 \end{aligned}$$

The states $|1\ \ 1\rangle$, $|1\ \ 0\rangle$, $|1\ \ -1\rangle$ form what are called the *triplet* states, corresponding to $s = 1$. Note that these are symmetric with respect to exchange of the two spins. The state $|0\ \ 0\rangle$ is known as the *singlet* state as it is antisymmetric (changes sign) upon exchange of the spins.

II. THE GENERAL PROBLEM

The problem of adding two arbitrary angular moment $\mathbf{J}_1$ and $\mathbf{J}_2$ amounts to finding a unitary transformation from the set of basis vectors defined by the tensor products of the individual eigenstates of J_1^2 and J_{1z} and J_2^2 and J_{2z} to the eigenstates of J^2 and J_z, where

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

The individual eigenstates of 1 and 2 satisfy

$$\begin{aligned} J_1^2|j_1\ \ m_1\rangle &= j_1(j_1+1)\hbar^2|j_1\ \ m_1\rangle \\ J_{1z}|j_1\ \ m_1\rangle &= m_1\hbar|j_1\ \ m_1\rangle \\ J_2^2|j_2\ \ m_2\rangle &= j_2(j_2+1)\hbar^2|j_2\ \ m_2\rangle \\ J_{2z}|j_2\ \ m_2\rangle &= m_2\hbar|j_2\ \ m_2\rangle \end{aligned}$$

We may also define raising and lowering operators according to

$$\begin{aligned} J_{1\pm}|j_1\ \ m_1\rangle &= \hbar\sqrt{j_1(j_1+1) - m_1(m_1\pm 1)}|j_1\ \ m_1\pm 1\rangle \\ J_{2\pm}|j_2\ \ m_2\rangle &= \hbar\sqrt{j_2(j_2+1) - m_2(m_2\pm 1)}|j_2\ \ m_2\pm 1\rangle \end{aligned}$$

We then define the tensor product basis vectors as

$$|j_1\ \ m_1;j_2\ \ m_2\rangle = |j_1\ \ m_1\rangle \bigotimes |j_2\ \ m_2\rangle \qquad m_1 = -j_1,...,j_1,\ \ m_2 = -j_2,...,j_2$$

and we seek a transformation to a set basis set denoted $|J\ \ M\rangle$ that satisfies

$$\begin{aligned} J^2|J\ \ M\rangle &= J(J+1)\hbar^2|J\ \ M\rangle \\ J_z|J\ \ M\rangle &= M\hbar|J\ \ M\rangle \\ J_\pm|J\ \ M\rangle &= \hbar\sqrt{J(J+1) - M(M\pm 1)}|J\ \ M\pm 1\rangle \end{aligned}$$

The method of obtaining the transformation is simply to expand the new basis vectors in terms of the old:

$$|J\ \ M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1\ \ m_1; j_2\ \ m_2\rangle\langle j_1\ \ m_1; j_2\ \ m_2|J\ \ M\rangle$$

The coefficients

$$\langle j_1\ \ m_1; j_2\ \ m_2|J\ \ M\rangle$$

are the general *Clebsch-Gordan* coefficients. In principle, they can be determined by the programmatic procedure outlined in the last section applied to the arbitrary angular momenta. Note that

$$\langle j_1\ \ m_1; j_2\ \ m_2|J\ \ M\rangle = 0$$

unless $m_1 + m + 2 = M$, which restricts the summations in the above expansion considerably.

Although a general formula exists for Clebsch-Gordon coefficients (see below), let us first examine some of the properties of the coefficients that are useful in constructing the unitary transformation:

1. The Clebsch-Gordon coefficients are real:

$$\langle j_1\ \ m_1; j_2\ \ m_2|J\ \ M\rangle = \langle J\ \ M|j_1\ \ m_1; j_2\ \ m_2\rangle$$

2. Orthogonality:

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle J\ \ M|j_1\ \ m_1; j_2\ \ m_2\rangle\langle j_1\ \ m_1; j_2\ \ m_2|J'\ \ M'\rangle = \delta_{JJ'}\delta_{MM'}$$

This can be seen by recognizing that

$$\langle J\ \ M|J'\ \ M'\rangle = \delta_{JJ'}\delta_{MM'}$$

so that if we insert an identity operator in the inner product in the form

$$I = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1\ \ m_1; j_2\ \ m_2\rangle\langle j_1\ \ m_1; j_2\ \ m_2|$$

the orthogonality relation results. A similar orthogonality relation is

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} \langle j_1\ \ m_1; j_2\ \ m_2|J\ \ M\rangle\langle J\ \ M|j_1\ \ m_1'; j_2\ \ m_2'\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'}$$

which can also be proved starting from

$$\langle j_1\ \ m_1; j_2\ \ m_2|j_1\ \ m_1'; j_2\ \ m_2'\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'}$$

and inserting identity in the form

$$I = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} J\ \ M\rangle\langle J\ \ M|$$

Note that the label J is not a fixed label like j_1 and j_2. This is because different total J values can result. The minimum value of J is clearly $|j_1 - j_2|$, while its maximum is $j_1 + j_2$, and we need to sum over these in the completeness relation.

3. Recursion relation:

$$\sqrt{J(J+1)-M(M\pm 1)}\langle j_1 \;\; m_1; j_2 \;\; m_2||J \;\; M\pm 1\rangle = \sqrt{j_1(j_1+1)-m_1(m_1\mp 1)}\langle j_1 \;\; m_1\pm 1; j_2 \;\; m_2|J \;\; M\rangle$$

$$+\sqrt{j_2(j_2+1)-m_2(m_2\mp 1)}\langle j_1 \;\; m_1; j_2 \;\; m_2\mp 1|J \;\; M\rangle$$

which can be derived starting from

$$J_\pm = J_{1\pm} + J_{2\mp}$$

and taking matrix elements of both sides between the new and old basis vectors:

$$\langle j_1 \;\; m_1; j_2 \;\; m_2|J_\pm|J \;\; M\rangle = \langle j_1 \;\; m_1; j_2 \;\; m_2|J_{1\pm}|J \;\; M\rangle + \langle j_1 \;\; m_1; j_2 \;\; m_2|J_{2\pm}|J \;\; M\rangle$$

On the left side, $J_\pm$ acts on $|J \;\; M\rangle$ to produce the term on the left in the recursion relation. On the right, the operators $J_{1\pm}$ and $J_{2\pm}$ operate to the left as $J_{1\pm}^\dagger$ and $J_{2\pm}^\dagger$. However,

$$J_{1\pm}^\dagger = J_{1\mp}$$
$$J_{2\pm}^\dagger = J_{2e\mp}$$

and hence produce the opposite action as the $J_\pm$ on the left.

Finally, the general formula for the Clebsch-Gordan coefficients is

$$\langle j_1 \;\; m_1; j_2 \;\; m_2|J \;\; M\rangle = \delta_{m_1+m_2,M}$$
$$\times\left[(2J+1)\frac{(s-2J)!(s-2j_2)!(s-2j_1)!}{(s+1)!}(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!\right]^{1/2}$$
$$\times\sum_\nu(-1)^\nu\frac{1}{\nu!(j_1+j_2-J-\nu)!(j_1-m_1-\nu)!(j_2+m_2-\nu)!(J-j_2+m_1+\nu)!(J-j_1-m_2+\nu)!}$$

where $s = j_1+j_2+J$, and the ν summation runs over all values for which all of the factorial arguments are greater than or equal to 0.

This formula is rather cumbersome to work with, so it is useful to deduce some special cases. These are as follows:

$$\langle j_1 \;\; j_1; j_2 \;\; j_2|J \;\; J\rangle = 1$$

i.

ii. if $m_1 = \pm j_1$ or $m_2 = \pm j_2$ and $M = \pm J$, then

$$\langle j_1 \;\; m_1; j_2 \;\; m_2|J \;\; J\rangle = \langle j_1 \;\; -m_1; j_2 \;\; -m_2|J \;\; -J\rangle$$

$$= (-1)^{j_1-m_1}\sqrt{\frac{(2J+1)!(j_1+j_2-J)!}{(j_1+j_2+J+1)!(J+j_1-j_2)!(J-j_1+j_2)!}}\sqrt{\frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!}}$$

iii. If $j_1+j_2 = J$,

$$\langle j_1 \;\; m_1; j_2 \;\; m_2|J \;\; M\rangle = \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!}}\sqrt{\frac{(J+M)!(J-M)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}$$

III. THE SIMPLE EXAMPLE REVISITED

Consider, again, the example of adding to spin-1/2 angular momenta. This time, we will use the Clebsch-Gordan coefficients directly to determine the unitary transformation. Start with the state $|1\ \ 1\rangle$. Expanding gives

$$|1\ \ 1\rangle = \sum_{m_1=-1/2}^{1/2}\sum_{m_2=-1/2}^{1/2}\left|\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right\rangle\left\langle\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right|1\ \ 1\rangle$$

Only one term gives $m_1+m_2=1$, which is clearly $m_1=1/2$ and $m_2=1/2$. Thus,

$$|1\ \ 1\rangle = \left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right|1\ \ 1\rangle$$

The Clebsch-Gordan coefficients, by special case i is just 1, so

$$|1\ \ 1\rangle = \left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle$$

as expected.

The state $|1\ \ 0\rangle$ is expanded as

$$|1\ \ 0\rangle = \sum_{m_1=-1/2}^{1/2}\sum_{m_2=-1/2}^{1/2}\left|\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right\rangle\left\langle\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right|1\ \ 0\rangle$$

This time, since $m_1+m_2=0$, two terms contribute, $m_1=1/2, m_2=-1/2$ and $m_1=-1/2, m_2=1/2$. Hence,

$$|1\ \ 0\rangle = \left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right|1\ \ 0\rangle + \left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right|1\ \ 0\rangle$$

However, since $j_1+j_2=1$, we can use special case iii, and we find

$$\left\langle\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right|1\ \ 0\rangle = \sqrt{\frac{1!1!}{2!}}\sqrt{\frac{1!1!}{1!0!0!1!}} = \frac{1}{\sqrt{2}}$$

$$\left\langle\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right|1\ \ 0\rangle = \sqrt{\frac{1!1!}{2!}}\sqrt{\frac{1!1!}{0!1!1!0!}} = \frac{1}{\sqrt{2}}$$

so that

$$|1\ \ 0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle + \left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle\right)$$

It is straightforward to show that $|1\ \ -1\rangle = \left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle$. The state $|0\ \ 0\rangle$ is expanded to give

$$|0\ \ 0\rangle = \sum_{m_1=-1/2}^{1/2}\sum_{m_2=-1/2}^{1/2}\left|\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right\rangle\left\langle\frac{1}{2}\quad m_1;\frac{1}{2}\quad m_2\right|0\ \ 0\rangle$$

Again there are two terms that contribute. However, to determine the two Clebsch-Gordan coefficients:

$$\left\langle\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right|0\ \ 0\rangle \qquad \text{and}\left\langle\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right|0\ \ 0\rangle$$

we can use special case ii, since $M=J$. In this case,

$$\left\langle\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right|0\ \ 0\rangle = (-1)^{\frac{1}{2}-\frac{1}{2}}\sqrt{\frac{1!1!}{2!0!0!}}\sqrt{\frac{1!0!}{0!1!}} = \frac{1}{\sqrt{2}}$$

$$\left\langle\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right|0\ \ 0\rangle = (-1)^{\frac{1}{2}+\frac{1}{2}}\sqrt{\frac{1!1!}{2!0!0!}}\sqrt{\frac{0!1!}{1!0!}} = -\frac{1}{\sqrt{2}}$$

Hence, the state is given by

$$|0\ \ 0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\quad\frac{1}{2};\frac{1}{2}\quad-\frac{1}{2}\right\rangle - \left|\frac{1}{2}\quad-\frac{1}{2};\frac{1}{2}\quad\frac{1}{2}\right\rangle\right)$$

IV. FINAL REMARKS

Note finally, that the number of basis vectors is generally given by $(2j_1+1)(2j_2+1)$, and that this number is equal to

$$\sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1)=(2j_1+1)(2j_2+1)$$

Also, sometimes one sees the so called "$3J$" symbols used instead of Clebsch-Gordan coefficients. This are denoted as

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_1 & M \end{pmatrix}$$

and are related to the Clebsch-Gordan coefficients by

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_1 & M \end{pmatrix} = (-1)^{j_1-j_2-M}\sqrt{2J+1}\langle j_1 \;\; m_1; j_2 \;\; m_2 | J \;\; M\rangle$$