

NAME and ID NUMBER:

There should be **19** pages to this exam, counting this cover sheet. Please check this exam NOW!

There are three pages of formulae/data, a periodic table (with molar masses), and a table of hydrogen-atom wave functions at the back of this exam. Books, notes, etc. are not permitted, however calculators are. Note that page 10 is intentionally left blank for you to use if you need additional space. Not all questions are equally difficult. Spending all your time on one question is generally a bad idea. Partial credit will be given but only if your answers can be deciphered. Therefore, make sure that your writing is neat and that your logic is clear, organized, and easy to follow. Unreadable answers will not be given the benefit of the doubt. Good luck.

GRADING

1. (10 points)
2. (30 points)
3. (20 points)
4. (20 points)
5. (20 points)

TOTAL: 100 points

Extra Credit:

1. (10 points)

Suppose you wanted to redefine the scale of relative masses such that all masses are defined relative to a scale in which gold has exactly one mass unit for each of its nucleons. Gold has $Z = 79$ and a mass number of $A = 197$. Using the fact that the mass of a proton is $1.6726216 \times 10^{-27}$ kg, the mass of a neutron is $1.6749272 \times 10^{-27}$ kg, and the mass of a carbon atom is $1.9264644 \times 10^{-26}$ kg, calculate the relative mass of carbon on the gold-based scale. Make any approximations you think are appropriate, but be sure to justify them.

2. (30 points)

a. (10 points)

A particle of mass m is in a one-dimensional box whose potential energy $V(x)$ is defined as follows: $V(x) = 0$ for $0 \leq x \leq L$ and $V(x) = \infty$ for $x < 0$ and $x > L$. Suppose the particle is in the $n = 3$ energy state. What is the probability that a measurement of the particle's position will yield a value between $x = 0.1L$ and $x = 0.2L$? The particle-in-a-box wave functions and energies are given by

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

b. (10 points)

What is the probability that a measurement of the particle's position will yield a value between $x = 0.3L$ and $x = 0.4L$ if the particle is in the $n = 3$ energy state?

c. (5 points)

In parts (a) and (b), the width of the interval for which you calculated the probability is the same, $0.1L$, in both cases. However, the *location* of the interval is clearly different in each case. Explain any similarity or difference between the probabilities you obtained in parts (a) and (b).

d. (5 points)

For a small interval Δx with endpoints at $x = a$ and $x = b$ with $\Delta x = b - a$, the probability that a measurement of the particle's position yields a value in this interval when the particle is an energy state with $n = 3$ can be *approximated* by the formula

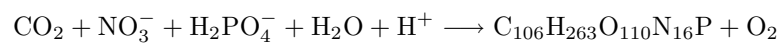
$$P(a \leq x \leq b) \approx \psi_3^2(x_0)\Delta x$$

where x_0 is the midpoint of the interval: $x_0 = (b - a)/2$. Use this formula to approximate the probabilities in parts (a) and (b). How accurate is this approximation for each probability?

3. (20 points)

Studies of biosynthesis are carried out on photosynthetic algae using trace amounts of the phosphorous anion H_2PO_4^- . When this anion is added to the aquatic environment of the algae, it is found that on the time scale of the experiment, *only* 50% of the anion is used for the synthesis of biomolecules, the sum total of which is denoted as $\text{C}_{106}\text{H}_{263}\text{O}_{110}\text{N}_{16}\text{P}$ (note that this chemical formula does not represent a single compound!). This maximum consumption of H_2PO_4^- is always observed independent of the amount originally added.

The *unbalanced* synthesis reaction is



a. (10 points)

Balance the reaction. Remember that both mass and charge must be balanced.

b. (10 points)

Suppose CO_2 , H_2O and H^+ are plentiful in the aquatic environment. In a given study, 5.0 g of H_2PO_4^- are added to the environment. Assuming 38.4 g of NO_3^- are already available. How many grams of “biomass” ($\text{C}_{106}\text{H}_{263}\text{O}_{110}\text{N}_{16}\text{P}$) will be produced in this case?

4. (20 points)

Suppose an atom with Z electrons is $(Z - 1)$ times ionized, leaving just one electron interacting with a nucleus of charge $+Ze$. The $1s$ wave function of this electron is

$$\psi_{1s}(r, \theta, \phi) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}$$

where a_0 is the Bohr radius.

a. (10 points)

What is the probability as a function of Z that a measurement of the electron's position yields a value of $r \geq a_0$ irrespective of the values of θ and ϕ ?

Hint: Before you do any integrals, check the information at the back of the exam!

b. (10 points)

Plot the probability you obtained in part (a) as a function of Z and explain the features of your graph, e.g. “The graph has a peak at a particular value of Z because...” or “The graph is strictly decreasing as a function of Z because...” or “The graph becomes negative at some point because...” or “The graph is always positive because...” Use as many such explanations as you think are relevant.

5. (20 points)

a. (5 points)

In classical mechanics, two potential energies $V(x)$ and $U(x)$ are considered the same if $U(x) = V(x) + C$, where C is a constant, because potential energies are always defined relative to some arbitrary energy value. In quantum mechanics, if two potential energies $U(x)$ and $V(x)$ differ only by a constant C , then the wave functions $\psi_n(x)$ associated with the two potential energies are the same, and only the energy levels E_n change. Starting with the Schrödinger equations for $U(x)$ and $V(x)$:

$$\left[\hat{K} + V(x) \right] \psi_n(x) = E_n \psi_n(x) \qquad \left[\hat{K} + U(x) \right] \psi_n(x) = E'_n \psi_n(x)$$

where \hat{K} is the kinetic energy operator and E'_n are the energy levels for the potential energy $U(x)$, show that if $U(x) = V(x) + C$, then

$$E'_n = E_n + C$$

b. (15 points)

Now suppose that $U(x)$ and $V(x)$ are truly different, meaning that there is *no* constant C for which $U(x) = V(x) + C$. In this problem, you will attempt to prove that the ground-state wave functions for $V(x)$ and $U(x)$ *must* be different. Let $\psi_1(x)$ be the ground state wave function for $V(x)$ with energy E_1 and $\psi'_1(x)$ be the ground-state wave function for $U(x)$ with energy E'_1 . These wave functions must satisfy the Schrödinger equations

$$\left[\hat{K} + V(x) \right] \psi_1(x) = E_1 \psi_1(x) \qquad \left[\hat{K} + U(x) \right] \psi'_1(x) = E'_1 \psi'_1(x)$$

In order to prove that $\psi_1(x)$ and $\psi'_1(x)$ cannot be the same, you will use a proof by contradiction. That is, prove that the assumption $\psi_1(x) = \psi'_1(x)$ leads to a logical contradiction, from which it must follow that $\psi_1(x) \neq \psi'_1(x)$.

Hint: Try substituting $\psi_1(x)$ into both of the above Schrödinger equations and then subtracting one of the resulting Schrödinger equations from the other. What do you get?

Extra Credit: (10 points)

The wave functions ψ_{nlm} of the hydrogen atom can be assigned definite values of the quantum numbers n , l and m . However, any p_x or p_y orbital, such as ψ_{2p_x} , ψ_{2p_y} , ψ_{3p_x} , ψ_{3p_y} , etc. has only definite values of n and l , while the value of m is ambiguous. Explain why.

POSSIBLY USEFUL INFORMATION

$$N_0 = 6.022 \times 10^{23} \text{ mol}^{-1}$$

$$k_B = 1.38066 \times 10^{-23} \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1} = 1.36262 \times 10^{-25} \text{ L} \cdot \text{atm} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$$

$$e = 1.60219 \times 10^{-19} \text{ C} \quad \epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \cdot \text{J}^{-1} \cdot \text{m}^{-1}$$

$$h = 6.6208 \times 10^{-34} \text{ J} \cdot \text{s} \quad \hbar = \frac{h}{2\pi} = 1.105457 \times 10^{-34} \text{ J} \cdot \text{s}$$

$$c = 2.99769 \times 10^8 \text{ m/s}$$

$$a_0 = 0.529177 \times 10^{-10} \text{ m}$$

Some conversion factors

$$1 \text{ \AA} = 10^{-8} \text{ cm} = 10^{-10} \text{ m}$$

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$$

$$1 \text{ Ry} = 2.18 \times 10^{-18} \text{ J}$$

Formulas

$$E_{\text{Coul}} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|} = \frac{q_1 q_2}{4\pi\epsilon_0 r} \quad \mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3} \mathbf{r}$$

$$KE = \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad A = \sum_{i=1}^N A_i p_i$$

$$A(x, t) = A_0 \cos\left(\frac{2\pi x}{\lambda} - 2\pi\nu t\right) \quad v = \lambda\nu \quad c = \lambda\nu$$

$$V(x) = \frac{1}{2} k (x - x_0)^2 \quad F = -k(x - x_0) \quad I(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

$$h\nu = \Phi + \frac{p^2}{2m_e} \quad \lambda = \frac{h}{p} \quad A(y) = \cos(S(y)) + i \sin(S(y)) = e^{iS(y)} \quad P(y) = \left| \sum_{\text{paths}} A_{\text{path}}(y) \right|^2$$

$$\Delta A \Delta B \geq \frac{1}{2} \hbar \quad \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \quad \Delta x \Delta p \geq \frac{1}{2} \hbar$$

$$E = \frac{p^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0 r} \quad dV = r^2 \sin\theta dr d\theta d\phi$$

$$E_n = -\frac{Z^2 e^4 m_e}{8\epsilon_0^2 \hbar^2} \frac{1}{n^2} = -\frac{Z^2}{n^2} \text{ (in Ry)} \quad a_0 = \frac{4\pi\epsilon_0 \hbar^2}{e^2 m_e}$$

$$h\nu = E_{n_i} - E_{n_f} \text{ (emission)} \quad h\nu = E_{n_f} - E_{n_i} \text{ (absorption)}$$

$$\hat{H}\psi(x) = E\psi(x) \quad p(x)dx = |\psi(x)|^2 dx \quad P(x \in [a, b]) = \int_a^b |\psi(x)|^2 dx$$

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x) \quad \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Mathematics

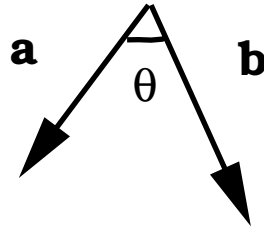


FIG. 1.

$$\mathbf{a} = (a_x, a_y, a_z)$$

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\mathbf{a} + \mathbf{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$|\mathbf{a} + \mathbf{b}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}}$$

$$\theta = \cos^{-1} \left[\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right]$$

Trigonometry

$$\cos^2(x) + \sin^2(x) = 1$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

$$e^{ix} = \cos(x) + i \sin(x)$$

Indefinite integrals

$$\int x^2 e^{-x} dx = -e^{-x} [x^2 + 2x + 2]$$

$$\int \sin(ax) \sin(bx) dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)}$$

$$\int \sin^2(ax) dx = \frac{1}{2}x - \frac{1}{4a} \sin(2ax)$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad \frac{dF}{dx} = f(x)$$