

# Simultaneous Equations - identification

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QII - Week 2

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## Exogeneity

- If we have two sets of variables, say  $\mathbf{y}$  and  $\mathbf{x}$ ,  $\mathbf{x}$  is (weakly) exogenous to (for)  $\mathbf{y}$  if
- We can estimate the parameters of the process determining  $\mathbf{y}$
- Conditional on the observed  $\mathbf{x}$
- That is, ignoring the stochastic process which determines  $\mathbf{x}$
- In symbols, we always have (by laws of conditional densities)

$$f_J(\mathbf{y}, \mathbf{x}) = f_C(\mathbf{y}|\mathbf{x}, \theta) f_M(\mathbf{x}|\theta) \quad (1)$$

- In the special case where the parameters that determine the marginal density of  $\mathbf{x}$  bear no relation to the parameters that determine the conditional density of  $\mathbf{y}$  given  $\mathbf{x}$ , so that

$$f_J(\mathbf{y}, \mathbf{x}) = f_C(\mathbf{y}|\mathbf{x}, \theta_1) f_M(\mathbf{x}|\theta_2) \quad (2)$$

where  $\theta_1$  and  $\theta_2$  are unrelated

- then  $\mathbf{x}$  is said to be (weakly) endogenous to  $\mathbf{y}$

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- Note this is for a specific model, so  $\mathbf{x}$  may be endogenous for some purposes and exogenous for others
- Trivial example: weight is exogenous to vote but endogenous to height
- In OLS we assume the independent variables are exogenous to the dependent variable
- Have to think about whether this is true
- Are attitudes about candidates exogenous to vote?
- That is, can we estimate vote conditional on attitudes assuming that whatever determines attitudes does not include anything about vote?

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## Identification

- Define the structure of a model as  $F(\mathbf{x}|\theta)$
- Two structures are observationally equivalent if  $F(\mathbf{x}|\theta_1) = F(\mathbf{x}|\theta_2)$
- In other words, there are two sets of parameters that say identical things about any data we might observe, so from that observed data we cannot get any evidence of which of the two parameter values is more supported by the data
- A parameter is IDENTIFIED if there are no other parameters that are observationally equivalent
- If not identified, then even with no error term and your favorite deity whispered some parameter values in your ear, you could say "but it equally well could be something else"
- eg multicollinearity
- say you are estimating  $y = a + bx + dx$
- favorite deity whispers  $a = 5, b = 3$
- you could reply why not  $a = 7, b = 1$

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- For concreteness, let us consider a 2 equation setup, so we have 2 endogenous (non-exogenous) variables,  $y_1$  and  $y_2$  for each person (perhaps vote and liberalism) and say three exogenous variables,  $x_1, x_2, x_3$ , say age, income and race.
- Suppose we think that  $y_1$  causes  $y_2$  and reverse, and both are functions of all the exogenous variables
- So the system (STRUCTURAL FORM)

$$y_1 = \beta_1 y_2 + \gamma_{1,1} x_1 + \gamma_{1,2} x_2 + \gamma_{1,3} x_3 + \epsilon_1 \quad (3)$$

$$y_2 = \beta_2 y_1 + \gamma_{2,1} x_1 + \gamma_{2,2} x_2 + \gamma_{2,3} x_3 + \epsilon_2 \quad (4)$$

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- We can also write more generally, for all the  $N$  observations

$$\mathbf{YB} + \mathbf{X}\mathbf{\Gamma} = \mathbf{E} \quad (5)$$

where if we have  $K$  endogenous variables and  $L$  exogenous,  $\mathbf{B}$  is  $K \times K$  and  $\mathbf{\Gamma}$  is  $L \times K$ ,  $\mathbf{Y}$  is  $N \times K$ ,  $\mathbf{X}$  is  $N \times L$  and  $\mathbf{E}$  is  $N \times K$ .

- (Looks a bit odd because the matrices have the observations on the rows and the  $K$  endog (or  $L$  exog) variables on the rows)
- The diagonal elements of  $\mathbf{B}$  must be one (why?)

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- We can use simple algebra to rewrite Eq. ?? having  $y_1$  (and the same for  $y_2$ ) as a linear function of the  $x$ 's and a new error term
- Or, assume  $\mathbf{B}$  is non-singular we get the REDUCED FORM

$$\mathbf{Y} + \mathbf{X}\mathbf{\Gamma}\mathbf{B}^{-1} = \mathbf{E}\mathbf{B}^{-1} \quad (6)$$

$$\mathbf{Y} = \mathbf{X}\mathbf{\Pi} + \mathbf{V} \quad (7)$$

where  $\mathbf{\Pi} = -\mathbf{\Gamma}\mathbf{B}^{-1}$  and  $\mathbf{V} = \mathbf{E}\mathbf{B}^{-1}$

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## Indirect least squares

- We can always estimate the reduced form by OLS, since it just has an exogenous  $y$  as a function of exogenous  $x$  (single  $y$ , all  $x$ )
- If only interested in this relationship, fine
- But if care about the impact of  $y_2$  on  $y_1$  then we need to go back to the structural form
- That is, we may care about the  $\mathbf{B}$  and  $\mathbf{\Gamma}$ , not just the  $\mathbf{\Pi}$

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- Can see identification in a number of ways
- One can see if a unique solution for  $\mathbf{B}$  and  $\mathbf{\Gamma}$  exists if we know  $\mathbf{\Pi}$  (or scalar analogue)
- Note that  $\mathbf{\Gamma}$  has  $L \times K$  unknowns,  $\mathbf{\Gamma}$  has  $K \times (K - 1)$  unknowns (since the coefficient on one parameter is one)
- But  $\mathbf{\Pi}$  has only  $L \times K$  parameters
- So we have more unknowns than equations, and so an infinite number of solutions are consistent with the reduced form parameters
- EVEN IF WE KNEW THOSE PARAMETERS PERFECTLY

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## Identification by zero restrictions

- A structural equation model will be identified if we can reduce the number of coefficients to estimate
- This is typically done by finding some that we believe to be zero
- This must be from belief, theory, or other *a priori* argument
- Since data cannot tell us anything about these restrictions
- If false, system not identified
- System can be underidentified, exactly identified or overidentified
- A number of ways to understand how many restrictions must be imposed

# How many restrictions

- Example of supply and demand
- Example adding an exogenous shock to supply
- Add another one to demand
- Counting moment restrictions
- Restrictions on adding up equations
- Result: an equation is identified if exactly as many zero restrictions on exogenous variables (in the system) as endogenous variables included in the equation
- If fewer restrictions, not identified (under identified)
- If more than necessary overidentified
- In latter case can test the overidentifying restrictions
- Other ways of identifying: restrictions on covariance of error
- Triangular systems