A. ~~Find the homogeneous solution for p_t^*~~

~~**Answer**: Form the homogeneous equation $p_t^* - (1 - \alpha)\, p_{t-1}^* = 0$.~~
~~The homogeneous solution is:~~

$$\boxed{p_t^* = A(1-\alpha)^t}$$

~~*where A* is an arbitrary constant and $(1-\alpha)$ is the characteristic root.~~

~~B. Use lag operators to find the particular solution. Check your answer by substituting your answer into the original difference equation.~~

~~**Answer**: The particular solution can be written as~~

~~$[\, 1 - (1-\alpha)L \,]\ p_t^* = \alpha p_{t-1}$~~

~~or $p_t^* = \alpha p_{t-1}/[\, 1 - (1-\alpha)L \,]$ so that~~

$$\boxed{p_t^* = \alpha[p_{t-1} + (1-\alpha)p_{t-2} + (1-\alpha)^2 p_{t-3} + \dots\,]}$$

~~To check the answer, substitute the particular solution into~~ the original difference equation

~~$\alpha[p_{t-1} + (1-\alpha)p_{t-2} + (1-\alpha)^2 p_{t-3} + \dots\,] = \alpha p_{t-1} + (1-\alpha)\alpha[p_{t-2} +$~~ $(1-\alpha)p_{t-3} + (1-\alpha)^2 p_{t-4} + \dots\,]$

~~It should be clear that the equati[illegible] holds as an identity.~~

3. Suppose that the money supply process has the form $m_t = m + \rho m_{t-1} + \varepsilon_t$ where m is a constant and $0 < \rho < 1$.

A. Show that it is possible to express m_{t+n} in terms of the known value m_t and the sequence $\{\varepsilon_{t+1}, \varepsilon_{t+2}, \dots, \varepsilon_{t+n})$.

Answer: One method is to use forward iteration. Updating the money supply process one period yields $m_{t+1} = m + \rho m_t + \varepsilon_{t+1}$. Update again to obtain

$$m_{t+2} = m + \rho m_{t+1} + \varepsilon_{t+2}$$
$$= m + \rho[m + \rho m_t + \varepsilon_{t+1}] + \varepsilon_{t+2} = m + \rho m + \varepsilon_{t+2} + \rho\varepsilon_{t+1} + \rho^2 m_t$$

Repeating the process for m_{t+3}

$$m_{t+3} = m + \rho m_{t+2} + \varepsilon_{t+3}$$

$$= m + \varepsilon_{t+3} + \rho[m + \rho m + \varepsilon_{t+2} + \rho\varepsilon_{t+1} + \rho^2 m_t]$$

For any period $t+n$, the solution is

$$\boxed{m_{t+n} = m(1 + \rho + \rho^2 + \rho^3 + \ldots + \rho^{n-1}) + \varepsilon_{t+n} + \rho\varepsilon_{t+n-1} + \ldots + \rho^{n-1}\varepsilon_{t+1} + \rho^n m_t}$$

B. Suppose that all values of ε_{t+i} for $i > 0$ have a mean value of zero. Explain how you could use your result in part A to forecast the money supply n-periods into the future.

Answer: The expectation of ε_{t+1} through ε_{t+n} is equal to zero. Hence, the expectation of the money supply n periods into the future is

$$\boxed{m(1 + \rho + \rho^2 + \rho^3 + \ldots + \rho^{n-1}) + \rho^n m_t}$$

As $n \to \infty$, the forecast approaches $m/(1-\rho)$.

~~4. Find the particular solutions for each of the following:~~

~~**i**. $y_t = a_1 y_{t-1} + \varepsilon_t + \beta_1\varepsilon_{t-1}$~~

~~**Answer**: Assuming $|a_1| < 1$, you can use lag operators to write the equation as $(1 - a_1L)y_t = \varepsilon_t + \beta_1\varepsilon_{t-1}$. Hence, $y_t = (\varepsilon_t + \beta_1\varepsilon_{t-1})/(1 - a_1L)$.~~

~~Now apply the expression $(1 - a_1L)^{-1}$ to each term in the numerator so that~~

~~$y_t = \varepsilon_t + a_1\varepsilon_{t-1} + (a_1)^2\varepsilon_{t-2} + (a_1)^3\varepsilon_{t-3} + \ldots + \beta_1[\varepsilon_{t-1} + a_1\varepsilon_{t-2} + (a_1)^2\varepsilon_{t-3} + \ldots]$~~

~~$\boxed{y_t = \varepsilon_t + (a_1+\beta_1)\varepsilon_{t-1} + a_1(a_1+\beta_1)\varepsilon_{t-2} + (a_1)^2(a_1+\beta_1)\varepsilon_{t-3} + (a_1)^3(a_1+\beta_1)\varepsilon_{t-4} + \ldots}$~~

~~If $a_1 = 1$, the improper form of the particular solution is:~~

~~$y_t = b_0 + \varepsilon_t + (1+\beta_1)\sum_{i=1}^{\infty}\varepsilon_{t-i}$~~

~~*where*: an initial condition is needed to eliminate the constant b_0 and the non-convergent sequence.~~

~~**ii**. $y_t = a_1 y_{t-1} + \varepsilon_{1t} + \beta\varepsilon_{2t}$~~

~~**Answer**: Write the equation as $y_t = \varepsilon_{1t}/(1-a_1L) + \beta\varepsilon_{2t}/(1-a_1L)$. Now, apply $(1 - a_1L)^{-1}$ to each term in the numerator so that~~

~~$y_t = \varepsilon_{1t} + a_1\varepsilon_{1t-1} + (a_1)^2\varepsilon_{1t-2} + (a_1)^3\varepsilon_{1t-3} + \ldots + \beta[\varepsilon_{2t} + a_1\varepsilon_{2t-1} + (a_1)^2\varepsilon_{2t-2} + (a_1)^3\varepsilon_{2t-3} + \ldots]$~~

~~Alternatively, you can use the Method of Undetermined Coefficients and write the challenge solution in the form~~

~~$y_t = \Sigma c_i\varepsilon_{1t-i} + \Sigma d_i\varepsilon_{2t-i}$~~
~~*where* the coefficients satisfy: $c_i = (a_1)^i$ and $d_i = \beta(a_1)^i$.~~

5. The *Unit Root Problem* in time-series econometrics is concerned with characteristic roots that are equal to unity. In order to preview the issue:

A. Find the homogeneous solution to each of the following.

i) $y_t = a_0 + 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - 1.5y_{t-1} + .5y_{t-2} = 0$. The homogeneous solution will take the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - 1.5A\alpha^{t-1} + 0.5A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation

$$\alpha^2 - 1.5\alpha + 0.5 = 0$$

The two characteristic roots are $\alpha_1 = 1$, $\alpha_2 = 0.5$. The linear combination of the two homogeneous solutions is also a solution. Hence, letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is

$$\boxed{A_1 + A_2(0.5)^t}$$

ii) $y_t = a_0 + y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - y_{t-2} = 0$. The homogeneous solution will take the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation $\alpha^2 - 1 = 0$. The two characteristic roots are $\alpha_1 = 1$, $\alpha_2 = -1$. The linear combination of the two homogeneous solutions is also a solution. Hence, letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is

$$\boxed{A_1 + A_2(-1)^t}$$

iii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - 2y_{t-1} + y_{t-2} = 0$. The homogeneous solution always takes the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - 2A\alpha^{t-1} + A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation

$$\alpha^2 - 2\alpha + 1 = 0$$

The two characteristic roots are $\alpha_1 = 1$, and $\alpha_2 = 1$; hence there is a repeated root. The linear combination of the two homogeneous solutions is also a solution. Letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is

$$\boxed{A_1 + A_2 t}$$

iv) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - y_{t-1} - 0.25y_{t-2} + 0.25y_{t-3} = 0$. The homogeneous solution always takes the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - A\alpha^{t-1} - 0.25A\alpha^{t-2} + 0.25A\alpha^{t-3} = 0$$

Next, divide by $A\alpha^{t-3}$ to obtain the characteristic equation

$$\alpha^3 - \alpha^2 - 0.25\alpha + 0.25 = 0$$

The three characteristic roots are $\alpha_1 = 1$, $\alpha_2 = 0.5$, and $\alpha_3 = -0.5$. The linear combination of the three homogeneous solutions is also a solution. Hence, letting A_1, A_2 and A_3 be three arbitrary constants, the complete homogeneous solution is

$$\boxed{A_1 + A_2(0.5)^t + A_3(-0.5)^t}$$

B. Show that each of the backward-looking particular solutions is not convergent.

i) $y_t = a_0 + 1.5y_{t-1} - .5y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - 1.5L + 0.5L^2)y_t = a_0 + \varepsilon_t$. Factoring the polynomial yields $(1 - L)(1 - 0.5L)y_t = a_0 + \varepsilon_t$. Although the expression $(a_0 + \varepsilon_t)/(1 - 0.5L)$ is convergent, $(a_0 + \varepsilon_t)/(1 - L)$ does not converge.

ii) $y_t = a_0 + y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 + L)y_t = a_0 + \varepsilon_t$. It is clear that neither $(a_0 + \varepsilon_t)/(1 - L)$ nor $(a_0 + \varepsilon_t)/(1 + L)$ converges.

iii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 - L)y_t = a_0 + \varepsilon_t$. Here there are two characteristic roots that equal unity. Dividing $(a_0 + \varepsilon_t)$ by either of the $(1 - L)$ expressions does not lead to a convergent result.

iv) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 - 0.5L)(1 + 0.5L)y_t = a_0 + \varepsilon_t$. The expressions $(a_0 + \varepsilon_t)/(1 + 0.5L)$ and $(a_0 + \varepsilon_t)/(1 - 0.5L)$ are convergent, but the expression $(a_0 + \varepsilon_t)/(1 - L)$ is not convergent.

C. Show that equation (i) can be written entirely in first-differences; i.e., $\Delta y_t = a_0 + .5\Delta y_{t-1} + \varepsilon_t$. Find the particular solution for Δy_t. [HINT: Define $y_t^* = \Delta y_t$ so that $y_t^* = a_0 - 0.5\, y_{t-1}^* + \varepsilon_t$. Find the particular solution for y_t^* in terms of the $\{\varepsilon_t\}$ sequence.]

Answer: Subtract y_{t-1} from each side of $y_t = a_0 + 1.5y_{t-1} - .5y_{t-2} + \varepsilon_t$ to obtain

$y_t - y_{t-1} = a_0 + 0.5y_{t-1} - .5y_{t-2} + \varepsilon_t$ so that

$$\begin{aligned}\Delta y_t &= a_0 + 0.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\ &= a_0 + 0.5\Delta y_{t-1} + \varepsilon_t\end{aligned}$$

The particular solution for $y_t^* = a_0 + 0.5\; y_{t-1}^* + \varepsilon_t$ is given by

$y_t^* = (a_0 + \varepsilon_t)/(1 - 0.5L)$ so that

$$\boxed{y_t^* = 2a_0 + \varepsilon_t + 0.5\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.125\varepsilon_{t-3} + \ldots.}$$

D. Similarly transform the other equations into their first-difference form. Find the backward-looking particular solution, if it exists, for the transformed equations.

ii) $y_t = a_0 + y_{t-2} + \varepsilon_t$,

Answer: Subtract y_{t-1} from each side to form $y_t - y_{t-1} = a_0 - y_{t-1} + y_{t-2} + \varepsilon_t$ or

$\Delta y_t = a_0 - \Delta y_{t-1} + \varepsilon_t$ so that

$$y_t^* = a_0 - y_{t-1}^* + \varepsilon_t$$

Note that the first difference Δy_t has characteristic root that is equal to -1. The proper form of the backward-looking solution does not exist for this equation. If you attempt the challenge solution $y_t^* = b_0 + \Sigma\alpha_i\varepsilon_{t-i}$, you find

$$b_0 + \alpha_0\varepsilon_t + \alpha_1\varepsilon_{t-1} + \alpha_2\varepsilon_{t-2} + \alpha_3\varepsilon_{t-3} + ... = a_0 - b_0 - \alpha_0\varepsilon_{t-1} - \alpha_1\varepsilon_{t-2} - \alpha_2\varepsilon_{t-3} - ... + \varepsilon_t$$

Matching coefficients on like terms yields

$b_0 = a_0 - b_0 \qquad \Rightarrow b_0 = a_0/2$
$\alpha_0 = 1$
$\alpha_1 = -\alpha_0 \qquad \Rightarrow \alpha_1 = -1$
and
$\alpha_i = (-1)^i$

In Part E, students are asked to solve an equation of this form with a given initial condition.

iii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$
Answer: Subtract y_{t-1} from each side to obtain $y_t - y_{t-1} = a_0 + y_{t-1} - y_{t-2} + \varepsilon_t$ so that

$$\Delta y_t = a_0 + \Delta y_{t-1} + \varepsilon_t$$

Using the definition of y_t^* it follows that $y_t^* = a_0 + y_{t-1}^* + \varepsilon_t$. Again, a proper form for the particular solution does not exist. The improper form is

$$y_t^* = a_0 t + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ...$$

Notice that the second difference $\Delta^2 y_t$ does have a convergent solution since

$$\Delta y_t^* = a_0 + \varepsilon_t$$

iv) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$
Answer: Subtract y_{t-1} from each side and note that $0.25y_{t-2} - 0.25y_{t-3} = 0.25\Delta y_{t-2}$ so that

$\Delta y_t = a_0 + 0.25\Delta y_{t-2} + \varepsilon_t$ or
$y_t^* = a_0 + 0.25\ y_{t-2}^* + \varepsilon_t$

Write the equation as $(1 - 0.25L^2)\ y_t^* = a_0 + \varepsilon_t$. Since $(1 - 0.25L^2) = (1 - 0.5L)(1 + 0.5L)$, it follows that

$$\boxed{y_t^* = (a_0 + \varepsilon_t)/[(1 - 0.5L)(1 + 0.5L)]}$$

E. Given the initial condition y_0, find the solution for: $y_t = a_0 - y_{t-1} + \varepsilon_t$.

Answer: You can use iteration or the Method of Undetermined Coefficients to verify that the solution is

$$\boxed{y_t = \sum_{i=1}^{t} (-1)^{i+t} \varepsilon_i + (-1)^t y_0 + \frac{a_0}{2}[1 - (-1)^t]}$$

Using the iterative method, $y_1 = a_0 + \varepsilon_1 - y_0$ and $y_2 = a_0 + \varepsilon_2 - y_1$ so that

$$y_2 = a_0 + \varepsilon_2 - a_0 - \varepsilon_1 + y_0 = \varepsilon_2 - \varepsilon_1 + y_0$$

Since $y_3 = a_0 + \varepsilon_3 - y_2$, it follows that $y_3 = a_0 + \varepsilon_3 - \varepsilon_2 + \varepsilon_1 - y_0$. Continuing in this fashion yields

$$y_4 = a_0 + \varepsilon_4 - y_3 = a_0 + \varepsilon_4 - a_0 - \varepsilon_3 + \varepsilon_2 - \varepsilon_1 + y_0 = \varepsilon_4 - \varepsilon_3 + \varepsilon_2 - \varepsilon_1 + y_0$$

To confirm the solution for y_t note that $(-1)^{i+t}$ is positive for even values of $(i+t)$ and negative for odd values of $(i+t)$, $(-1)^t$ is positive for even values of t, and $(a_0/2)[1 - (-1)^t]$ equals zero when t is even and a_0 when t is odd.

6. A researcher estimated the following relationship for the inflation rate (π_t):

$$\pi_t = -.05 + 0.7\pi_{t-1} + 0.6\pi_{t-2} + \varepsilon_t$$

A. Suppose that in periods 0 and 1, the inflation rate was 10% and 11%, respectively. Find the homogeneous, particular, and general solutions for the inflation rate.

Answer: The homogeneous equation is $\pi_t - 0.7\pi_{t-1} - 0.6\pi_{t-2} = 0$. Try the challenge solution $\pi_t = A\alpha^t$, so that the characteristic equation is

$$A\alpha^t - 0.7A\alpha^{t-1} - 0.6A\alpha^{t-2} = 0 \text{ or}$$
$$\alpha^2 - 0.7\alpha - 0.6 = 0$$

The characteristic roots are: $\alpha_1 = 1.2$, $\alpha_2 = -0.5$. Thus, the homogeneous solution is

$$\boxed{\pi_t = A_1(1.2)^t + A_2(-0.5)^t}$$

The backward-looking particular solution is explosive. Try the challenge solution: $\pi_t = b + \Sigma b_i\varepsilon_{t-i}$. For this to be a solution, it must satisfy

$$\begin{aligned} b + b_0\varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + b_3\varepsilon_{t-3} + \ldots &= -0.05 + 0.7(b + b_0\varepsilon_{t-1} + b_1\varepsilon_{t-2} + b_2\varepsilon_{t-3} + b_3\varepsilon_{t-4} + \ldots) \\ &\quad + 0.6(b + b_0\varepsilon_{t-2} + b_1\varepsilon_{t-3} + b_2\varepsilon_{t-4} + b_3\varepsilon_{t-5} + \ldots) + \varepsilon_t \end{aligned}$$

Matching coefficients on like terms yields:

$b = -0.05 + 0.7b + 0.6b \qquad \Rightarrow b = 1/6$
$b_0 = 1$
$b_1 = 0.7b_0 \qquad \Rightarrow b_1 = 0.7$
$b_2 = 0.7b_1 + 0.6b_0 \qquad \Rightarrow b_2 = 0.49 + 0.6 = 1.09$

All successive values for b_i satisfy the explosive difference equation

$$\boxed{b_i = 0.7b_{i-1} + 0.6b_{i-2}}$$

If you continue in this fashion, the successive values of the b_i are:
$b_3 =1.183$; $b_4 = 1.4821$; $b_5 = 1.74727$; $b_6 = 2.11235$; $b_7 = 2.527007$...

Note that the forward-looking solution is not satisfactory here unless you are willing to assume perfect foresight. However, this is inconsistent with the presence of the error term. (After all, the regression would not have to be estimated if everyone had perfect foresight.) The point is that the forward-looking solution expresses the current inflation rate in terms of the future values of the $\{\varepsilon_t\}$ sequence. If $\{\varepsilon_t\}$ is assumed to be a white-noise process, it does not make economic sense to posit that the current inflation rate is determined by the future realizations of $\varepsilon_{t+i.}$

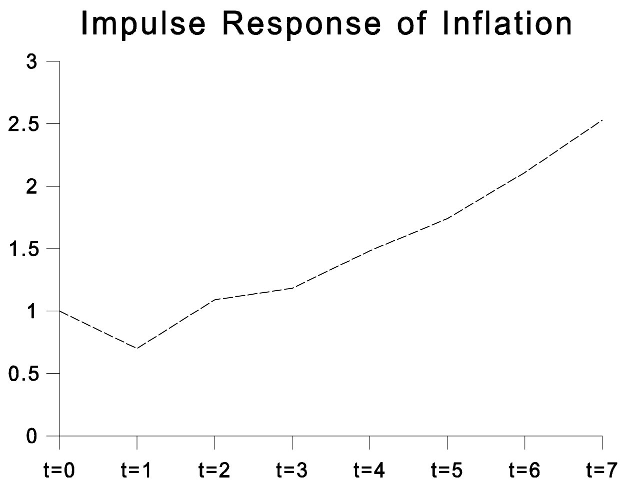
Although the backward-looking particular solution is not convergent, imposing the initial conditions on the particular solution yields finite values for all π_t (as long as t is finite). Consider the general solution

$$\pi_t = 1/6 + \varepsilon_t + 0.7\varepsilon_{t-1} + b_2\varepsilon_{t-2} + ... + b_{t-2}\varepsilon_2 + b_{t-1}\varepsilon_1 + b_t\varepsilon_0 + b_{t+1}\varepsilon_{-1} + ... + A_1(1.2)^t + A_2(-0.5)^t$$

For $t = 0$ and $t = 1$:

$0.10 = 1/6 + \varepsilon_0 + 0.7\varepsilon_{-1} + b_2\varepsilon_{-2} + ... + A_1 + A_2$
$0.11 = 1/6 + \varepsilon_1 + 0.7\varepsilon_0 + b_2\varepsilon_{-1} + ... + A_1(1.2) + A_2(-0.5)$

These last two equations define A_1 and A_2. Inserting the solutions for A_1 and A_2 into the general solution for π_t eliminates the arbitrary constants.



B. Discuss the shape of the impulse response function. Given that the U.S. is not headed for runaway inflation, why do you believe that the researcher's equation is poorly estimated?

> **Answer**: The impulse response function is given by the $\{b_i\}$ sequence. The impact of an ε_t shock on the rate of inflation is 1. Only 70% of this initial effect remains for one period. After this one-time decay, the effect of the ε_t shock on π_{t+2}, π_{t+3}, ... explodes. You can see the impulse response function in the accompanying chart. The impulse responses imply that the inflation rate will grow exponentially. Given that there will not be runaway inflation, we would want to disregard the estimated model.

~~7. Consider the stochastic process: $y_t = a_0 + a_2 y_{t-2} + \varepsilon_t$.~~

~~A. Find the homogeneous solution and determine the stability condition.~~

> ~~**Answer**: The homogeneous solution has the form $y_t = A\alpha^t$. Form the characteristic equation by substitution of the challenge solution into the original equation, so that~~
>
> ~~$A\alpha^t - a_2 A\alpha^{t-2} = 0$ so that $\alpha^2 = a_2$.~~
>
> ~~The two characteristic roots are $\alpha_1 = \sqrt{a_2}$ and $\alpha_2 = -\sqrt{a_2}$. The stability condition is for a_2 to be less than unity in absolute value.~~

~~B. Find the particular solution using the Method of Undetermined Coefficients.~~

> ~~**Answer**: Try the challenge solution $y_t = b + \Sigma b_i \varepsilon_{t-i}$. For this to be a solution, it must satisfy~~
>
> ~~$b + b_0\varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + b_3\varepsilon_{t-3} + ... = a_0 + a_2(b + b_0\varepsilon_{t-2} + b_1\varepsilon_{t-3} + b_2\varepsilon_{t-4} + b_3\varepsilon_{t-5} + ...) + \varepsilon_t$~~
>
> ~~Matching coefficients on like terms~~
>
> ~~$b = a_0 + a_2 b$~~ ~~$\Rightarrow b = a_0/(1-a_2)$~~