Notes on Simulation

Remember that there are 2 types of uncertainty we deal with:

Estimation Uncertainty: this is our uncertainty about what the true parameters of the model are. We can think of it as being caused by small samples, if N were infinity (not the entire sample, infinity!), then there would be no estimation uncertainty.

Fundamental Uncerainty: this is the stochastic component of the real world, it is the disturbance ϵ, so the measure of fundamental uncertainty is σ^2, the variance of ϵ.

Just Notation:

$$E[Y|X_i] = E[Y_i]$$

Note the following basics:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \tag{1}$$
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \tag{2}$$

$$\begin{aligned} E[Y_i] &= E[\beta_0 + \beta_1 X_i + \epsilon_i] \\ &= E[\beta_0 + \beta_1 X_i] \\ &= \beta_0 + \beta_1 X_i \end{aligned} \tag{3}$$

$$\textbf{So}: \; E[Y|X_i] = \beta_0 + \beta_1 X_i$$

$$\textbf{Important}!: \; E[Y_i] \neq \hat{\beta}_0 + \hat{\beta}_1 X_i \tag{4}$$

Note that we already know how to compute a confidence interval around β:

$$\hat{\beta} \sim \mathbf{N}(\beta, \mathbf{Var}(\hat{\beta}))$$

So we can now make a probabilistic statement about β:

$$\mathbf{Pr}(\beta - 1.96\sigma_{\hat{\beta}} < \hat{\beta} < \beta + 1.96\sigma_{\hat{\beta}}) = .95$$

We subtract $\hat{\beta}$ from all sides of the inequality, and we subtract β from all sides of the inequality:

$$\mathbf{Pr}(-\hat{\beta} - 1.96\sigma_{\hat{b}eta} < -\beta < -\hat{\beta} + 1.96\sigma_{\hat{\beta}}) = .95$$

Now multiply by -1 which flips the inequalities:

$$\mathbf{Pr}(\hat{\beta} - 1.96\sigma_{\hat{\beta}} < \beta < \hat{\beta} + 1.96\sigma_{\hat{\beta}}) = .95$$

So to get the confidence interval about "the expected value of Y given X_i", we need to look at equation (4) above. And we note that:

$$\mathbf{E}[\hat{Y}] \;=\; \mathbf{Y}$$

But obviously we don't know the variance of the truth. So we do what we always do. We know:

$$\hat{Y} \;\sim\; \mathbf{N}(\mathbf{E}[Y], \mathbf{Var}(\hat{Y}))$$

So we can now make a probabilistic statement that contains the expectation of Y:

$$\mathbf{Pr}(\mathbf{E}[Y] \;-\; 1.96\sigma_{\hat{Y}} \;<\; \hat{Y} \;<\; \mathbf{E}[Y] \;+\; 1.96\sigma_{\hat{Y}}) \;=\; .95$$

We add $-\hat{Y} \;-\; \mathbf{E}[Y]$ to all sections of the inequality:

$$\mathbf{Pr}(-\hat{Y} - 1.96\sigma_{\hat{Y}} \;<\; -\mathbf{E}[Y] \;<\; -\hat{Y} + 1.96\sigma_{\hat{Y}}) \;=\; .95$$

Now multiply by -1:

$$\mathbf{Pr}(\hat{Y} \;+\; 1.96\sigma_{\hat{Y}} \;>\; \mathbf{E}[Y] \;>\; \hat{Y} \;-\; 1.96\sigma_{\hat{Y}}) \;=\; .95$$

Re-arranging terms:

$$\mathbf{Pr}(\hat{Y} \ - \ 1.96\sigma_{\hat{Y}} \ < \ \mathbf{E}[Y] \ < \ \hat{Y} \ + \ 1.96\sigma_{\hat{Y}}) \ = \ .95$$

So now since we know $\hat{Y}$ what we need is $\sigma_{\hat{Y}}$. That is easy:

$$\hat{Y}_i \ = \ \hat{\beta}_0 \ + \ \hat{\beta}_1 X_i$$

$$\begin{aligned}\mathbf{Var}(\hat{Y}_i) \ = \ \mathbf{Var}(\hat{\beta}_0) \ + \ X_i^2 \ \mathbf{Var}(\hat{\beta}_1) \\ + \ 2 \ X_i \ \mathbf{COV}(\hat{\beta}_0, \ \hat{\beta}_1)\end{aligned}$$

And in OLS, we know:

$$\mathbf{VAR}(\hat{\beta}) \ = \ \sigma^2 (X'X)^{-1} \tag{5}$$

But, in cases more complicated than simple linear models such as OLS, we use simulation.

1 To Compute Confidence Intervals:

1. Estimate the model.
2. This produces estimates of the parameters (β), and the variance-covariance matrix of the parameters (V).
3. Draw a set of values of the parameters from a multivariate normal distribution: $N(\hat{\beta}, \hat{V})$.
4. Compute predicted values of the quantity of interest for:
   (a) a specified respondent or case
   (b) the entire sample
   (c) or, some relevant subsample.
5. Repeat the above procedure N times where N is a big number (500 or 1000 is generally realistic).
6. The sampling distribution of the predicted probabilities will give you the confidence interval.

See "Clarify"; URL = http://gking.harvard.edu

2 First Differences

$$Y_i = (\beta' X_i) \tag{6}$$

Say: we want to know what happens if X_{ik} were to change to $\tilde{X}_{ik}$; say X_{ik} were to increase by z units. [X_{ik} is a particular element of the vector X_i.]

1. Estimate the model.
2. This produces estimates of the parameters (β), and the variance-covariance matrix of the parameters (V).
3. Draw a set of N values of the parameters from a multivariate normal distribution: $N(\hat{\beta}, \hat{V})$.
4. For each of the N values of $\hat{\beta}$:
   - Compute: $\hat{Y}_i = \hat{\beta}' X_i$
   - Compute: $\tilde{X}_{ik} = X_{ik} + z$
   - Compute: $\tilde{Y}_i = \hat{\beta}' \tilde{X}_i$
   - Compute: $\tilde{Y}_i - \hat{Y}_i$

The last difference is the quantity of interest.

We could also compute this for **everyone** in the sample, and then compute the mean of $\tilde{Y}_i - \hat{Y}_i$; and get the effect of **all** respondents changing their taste on characteristic k by z units.